SPECIAL TOPICS IN ELEMENTARY PARTICLE PHYSICS

Physics 481

Problem Set #4 - Due Tuesday, October 30, 2007.

1. A search for $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ oscillations will be carried out by the NOνA (NuMI Off-Axis ν_e Appearance Experiment) Collaboration, utilizing neutrinos of average energy several GeV from Fermilab to a detector in northern Minnesota, a distance of 810 km. (A description of the experiment may be found at `http://www-nova.fnal.gov/`.) The neutrinos will pass through the mantle of the Earth, which may be assumed to be of constant density $\rho = 2.8$ g·cm^{-3}. Find the energy $E_\nu \equiv E_g$ for which the resonance parameter $x = 1$, where

$$x \equiv \frac{2\sqrt{2} G_F N_e E_\nu}{\Delta m^2} , \tag{1}$$

with N_e denoting the electron density. You may take $\Delta m^2 = 2.4 \times 10^{-3}$ eV^2.

Answer: The neutrino energy for $x = 1$ is

$$E_\nu = \frac{\Delta m^2}{2\sqrt{2} G_F N_e} . \tag{2}$$

The Fermi constant G_F is 1.16637×10^{-5} GeV^{-2}, while

$$N_e = (Z/A)\rho N_A = 8.43(2Z/A) \times 10^{23} \text{ cm}^{-3} . \tag{3}$$

To convert this to energy units we multiply it by

$$(\hbar c)^3 = (0.197 \text{ GeV} \cdot 10^{-13} \text{ cm})^3 = 7.645 \times 10^{-42} \text{ GeV}^3 \text{ cm}^3 , \tag{4}$$

so $N_e = 6.45 \times 10^{-18}$ GeV^3 and

$$E_\nu = \frac{(2Z/A) \cdot 2.4 \times 10^{-21} \text{ GeV}^2}{2\sqrt{2} \cdot 1.1663710^{-5} \text{ GeV}^{-2} \cdot 6.45 \times 10^{-18} \text{ GeV}^3} = 11.3 \cdot (2Z/A) \text{ GeV} . \tag{5}$$

2. In the limit of small θ_{13} and for maximal ν_μ–ν_τ mixing, one can write the PMNS matrix to first order in θ_{13} as

$$U_{PMNS} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & \theta_{13} \\ 0 & 1 & 0 \\ -\theta_{13} & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{6}$$

Using the form of U_{PMNS} obtained by multiplying out these matrices, calculate the probabilities $P_{\rm vac}(\nu_\mu \to \nu_e)$ and $P_{\rm vac}(\bar{\nu}_\mu \to \bar{\nu}_e)$ for detecting electron neutrinos or antineutrinos in the NOνA Detector if the oscillations were taking place in vacuum. How are these probabilities affected by passage through matter?

Answer: Performing the matrix multiplication, one has

$$U_{PMNS} \simeq \begin{bmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \theta_{13} \\ -\sqrt{\frac{1}{6}} + \frac{\theta_{13}}{\sqrt{3}} & \sqrt{\frac{1}{3}} + \frac{\theta_{13}}{\sqrt{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} - \frac{\theta_{13}}{\sqrt{3}} & \sqrt{\frac{1}{3}} - \frac{\theta_{13}}{\sqrt{3}} & \sqrt{\frac{1}{2}} \end{bmatrix} . \tag{7}$$

Here you may regard θ_{13} as real and small. The initial neutrinos are

$$\begin{aligned} \nu_\mu^{\rm init} &= \left(-\sqrt{\frac{1}{6}} + \frac{\theta_{13}}{\sqrt{3}}\right)\nu_1 + \left(\sqrt{\frac{1}{3}} + \frac{\theta_{13}}{\sqrt{3}}\right)\nu_2 - \sqrt{\frac{1}{2}}\nu_3 \\ &= \left(\frac{\nu_2}{\sqrt{3}} - \frac{\nu_1}{\sqrt{6}}\right) + \theta_{13}\left(\frac{\nu_1}{\sqrt{3}} + \frac{\nu_2}{\sqrt{6}}\right) - \frac{\nu_3}{\sqrt{2}} . \end{aligned} \tag{8}$$

Now, ν_i evolves as $e^{-im_i t}$, where t is proper time. For $L = 810$ km, $E = 3$ GeV, and $\Delta m_{12}^2 = 8 \times 10^{-5}$ GeV^2, we have

$$\frac{1.27 \Delta m_{12}^2 L}{E} = \frac{(1.27)(8 \times 10^{-5})(810)}{3} = 2.7 \times 10^{-2} , \tag{9}$$

so ν_1 and ν_2 may be regarded as remaining in phase with one another over the path between Fermilab and the NOνA detector, and they may be regarded as evolving with a common phase factor $e^{-i\bar{m}_{12}t}$, where $\bar{m}_{12} \equiv (m_1 + m_2)/2$. Henceforth we shall regard Δm_{32}^2 as $m_3^2 - \bar{m}_{12}^2$.

The combinations of mass eigenstates in Eq. (8) may be expressed in terms of flavor eigenstates, to appropriate order in θ_{13}, as

$$\frac{\nu_2}{\sqrt{3}} - \frac{\nu_1}{\sqrt{6}} \simeq \frac{\nu_\mu + \nu_\tau}{2} , \quad \frac{\nu_1}{\sqrt{3}} + \frac{\nu_2}{\sqrt{6}} \simeq \frac{\nu_e}{\sqrt{2}} , \quad \nu_3 \simeq \frac{\nu_\tau - \nu_\mu}{\sqrt{2}} + \theta_{13} , \tag{10}$$

so

$$\nu_\mu^{\rm init} \simeq \left[\frac{\nu_\mu + \nu_\tau}{2} + \theta_{13}\frac{\nu_e}{\sqrt{2}}\right] + \left[\frac{\nu_\mu - \nu_\tau}{2} - \theta_{13}\frac{\nu_e}{\sqrt{2}}\right] . \tag{11}$$

In this equation the first term in square brackets evolves as $e^{-i\bar{m}_{12}t}$ while the second evolves as $e^{-im_3 t}$. Hence the amplitude for detection of ν_e at a proper time t due to vacuum oscillations is

$$\langle \nu_e(t) | \nu_\mu^{\rm init} \rangle \simeq \frac{\theta_{13}}{\sqrt{2}}(e^{-i\bar{m}_{12}t} - e^{-im_3 t}) \tag{12}$$

so the probability of detecting $\nu_\mu \to \nu_e$ or $\bar{\nu}_\mu \to \bar{\nu}_e$ (they are the same in vacuum if θ_{13} is real) is

$$P_{\rm vac}(\nu_\mu \to \nu_e) = |\langle \nu_e(t) | \nu_\mu^{\rm init} \rangle|^2 \simeq 2\theta_{13}^2 \sin^2(\Delta m_{32}\ t/2) = 2\theta_{13}^2 \sin^2 \Delta_{\rm atm} \tag{13}$$

with $\Delta_{\rm atm} \equiv (1.27 \Delta m^2_{32} L/E)$, Δm^2_{32} in eV^2, L in km, and E in GeV. A more precise expression (see the NOνA documents referenced in the above web site) is

$$P_{\rm vac}(\nu_\mu \to \nu_e) = \sin^2\theta_{23} \sin^2 2\theta_{13} \sin^2 \Delta_{\rm atm} \ . \tag{14}$$

At the peak of an oscillation, since $\sin^2\theta_{23} \simeq 1/2$,

$$P_{\rm vac}(\nu_\mu \to \nu_e) \simeq \frac{1}{2}\sin^2 2\theta_{13} = 2.5\% \left(\frac{\sin^2 2\theta_{13}}{0.05}\right) \ . \tag{15}$$

(The current CHOOZ limit is $\sin^2 2\theta_{13} < 0.16$.) The first peak occurs when $\Delta_{\rm atm} = \pi/2$ and hence

$$E = 1.57 \text{ GeV} \left(\frac{\Delta m^2_{32}}{2.4 \times 10^{-3} \text{ eV}^2}\right)\left(\frac{L}{810 \text{ km}}\right) \ . \tag{16}$$

For oscillations in matter, we may use the usual transformations

$$\Delta m^2_M = \Delta m^2_{32} \sqrt{\sin^2 2\theta_{13} + (\cos 2\theta_{13} - x)^2} \ , \tag{17}$$

$$\sin 2\theta_M = \frac{\sin 2\theta_{13}}{\sqrt{\sin^2 2\theta_{13} + (\cos 2\theta_{13} - x)^2}} \ , \tag{18}$$

which may be approximated for small θ_{13} and x as

$$\Delta m^2_M \simeq \Delta m^2_{32} \ , \quad \sin 2\theta_M \simeq (2\theta_{13})^2 (1 \pm 2x) \ , \tag{19}$$

where the $+$ sign is for $\nu_\mu \to \nu_e$ and the $-$ sign is for $\bar\nu_\mu \to \bar\nu_e$. One sees that the result is sensitive to the sign of x and hence to the sign of Δm^2_{32}.