1. Show that one can measure a neutrino mass via a “missing-mass” technique (e.g., in the decay $\pi^+ \rightarrow \mu^+ + \nu_\mu$), or an oscillation technique, but not both. [This is a nice application of the Uncertainty Principle. See, e.g., B. Kayser, Phys. Rev. D24, 110 (1981).]

**ANSWER:** In order to see oscillations, one must specify $\Delta m^2 t/(4\hbar) = \Delta m^2 L/(4\hbar)$ to an accuracy much less than 1, where $L$ is the path length. (Here we have set $c = 1$.) That is, we need

$$\frac{\Delta m^2 \Delta L}{4\hbar} \ll 1 \quad \text{or} \quad \frac{\Delta m^2}{4\hbar} \ll \frac{\hbar}{\Delta L} < \Delta \rho \quad ,$$

(1)

where in the last step we have used the uncertainty principle. That is, the need to specify the path length has introduced an uncertainty in the momentum. But the last equation implies

$$\Delta m^2 \ll 2\Delta(p^2) \quad ,$$

(2)

i.e., the uncertainty in $p^2$ is larger than the squared mass difference which we want to measure. This then prevents us from determining the squared mass difference from the kinematic relation $m^2 = E^2 - p^2$ (assuming a known value of $E$). A more elegant discussion of this result can be found in the article by Kayser mentioned above.

2. It has been proposed to build an “off-axis” neutrino beam in order to study $\nu_\mu \rightarrow \nu_e$ oscillations. When a pion of laboratory energy $E_\pi$ decays to muon and a neutrino which makes an angle $\theta$ in the laboratory with respect to the pion, the energy $E_\nu$ of the neutrino is a function of $E_\pi$ and $\theta$. Calculate $E_\nu(E_\pi, \theta)$ in the limit $\theta \ll 1$ and $E_\pi \gg m_\pi$ and show that for a fixed value of $\theta$, $E_\nu$ attains a maximum for some value of $E_\pi$. (This means that the neutrino energy spectrum from a broad-band neutrino beam will show a peak at its maximum value, since $E_\nu$ is least sensitive to $E_\pi$ there.) Illustrate this behavior for the cases $\theta = 15, 30$ mr with a crude sketch and give the corresponding values of $E_\pi$ and $E_\nu^{\max}$.

**ANSWER:** Use 4-momentum conservation: $p_\pi - p_\nu = p_\mu$ and take the invariant square:

$$\left(p_\pi - p_\nu\right)^2 = p_\mu^2 = m_\mu^2 - m_\pi^2 - 2E_\pi E_\nu (1 - \beta_\pi \cos \theta) \quad ,$$

(3)

where $E_\pi$ and $E_\nu$ are the pion and neutrino lab energies, and $\beta_\pi$ is the pion velocity in the lab. Solving for $E_\nu$, one finds

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2E_\pi (1 - \beta_\pi \cos \theta)} \quad .$$

(4)
Now one makes the small-$\theta$, large-pion-energy approximation:

$$\beta_\pi = \sqrt{1 - \gamma_\pi^{-2}} \simeq 1 - \frac{m_\pi^2}{2E_\pi^2} , \quad \cos \theta \simeq 1 - \frac{\theta^2}{2} \quad (5)$$

and keeps leading terms in

$$1 - \beta_\pi \cos \theta \simeq \frac{m_\pi^2}{2E_\pi^2} + \frac{\theta^2}{2} \quad (6)$$

so that

$$E_\nu \simeq \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 / E_\pi + \theta^2 E_\pi} \quad (7)$$

This reaches a maximum for any $\theta$ when $E_\pi = m_\pi / \theta$.

The behavior is illustrated in the Figure. If $\theta = 15 \text{ mr (solid)}$, the maximum $E_\nu$ is 1.97 GeV, attained for $E_\pi = 9.3$ GeV, while for $\theta = 30 \text{ mr (dashed)}$, the maximum $E_\nu$ is 0.99 GeV, attained for $E_\pi = 4.7$ GeV. For small $E_\pi$ the neutrino energy is

$E_\nu \simeq E_\pi[1 - (m_\mu / m_\pi)^2]$ (dotted line).

![Graph showing the relation between $E_\pi$ and $E_\nu$.](image)