This exercise concerns the “U(1) problem” noted by S. Weinberg, Phys. Rev. D 11, 3583 (1975). Current algebra is employed to parametrize the pseudoscalar meson masses as follows in the absence of an axial U(1) anomaly:

\[ m_\pi^2 = c \left( \epsilon_0 + \frac{\epsilon_8}{\sqrt{2}} \right), \quad m_K^2 = c \left( \epsilon_0 - \frac{\epsilon_8}{2\sqrt{2}} \right), \]

and the physical \( \eta \) and \( \eta' \) are the eigenvalues of the 2 \( \times \) 2 matrix

\[ \mathcal{M}^2 = \begin{bmatrix} \mathcal{M}_{88}^2 & \mathcal{M}_{80}^2 \\ \mathcal{M}_{08}^2 & \mathcal{M}_{00}^2 \end{bmatrix}, \]

where

\[ \mathcal{M}_{88}^2 \equiv c \left( \epsilon_0 - \frac{\epsilon_8}{\sqrt{2}} \right), \quad \mathcal{M}_{80}^2 = \mathcal{M}_{08}^2 \equiv \frac{c}{z} \epsilon_8, \quad \mathcal{M}_{00}^2 \equiv \frac{c}{z^2} \epsilon_0. \]

Here \( z = F_0/F_\pi \) is the ratio of the SU(3)-singlet decay constant \( F_0 \) to the pion decay constant \( F_\pi = 93 \, \text{MeV} \).

(a) Express the matrix \( \mathcal{M}^2 \) in terms of \( m_\pi^2, m_K^2, \) and \( z \).

\textbf{Answer:} Algebraic combinations of Eqs. (1) and (3) lead to

\[ \mathcal{M}^2 = \begin{bmatrix} \frac{1}{3} (4m_K^2 - m_\pi^2) & -\frac{2\sqrt{2}}{3z} (m_K^2 - m_\pi^2) \\ -\frac{2\sqrt{2}}{3z} (m_K^2 - m_\pi^2) & \frac{1}{3z} (2m_K^2 + m_\pi^2) \end{bmatrix}. \]

(b) Show (either numerically or analytically) that the matrix always has at least one eigenvalue less than \( 3m_\pi^2 \).

\textbf{Answer:} Writing \( \mathcal{M}_{88} = \alpha m_\pi^2, \mathcal{M}_{08} = \beta m_\pi^2/z, \mathcal{M}_{00} = \gamma m_\pi^2/z^2, \) and \( r = m_K^2/m_\pi^2 = 13.05, \) the eigenvalues in units of \( m_\pi^2 \) are

\[ \lambda_\pm = \frac{1}{2} \left( \alpha + \frac{\gamma}{z^2} \right) \pm \frac{1}{4} \left( \frac{\alpha}{z^2} - \frac{\gamma}{z^2} \right)^2 + \frac{\beta^2}{z^2}, \]

where \( \alpha = (4r - 1)/3, \beta = -2\sqrt{2}(r - 1)/3, \) and \( \gamma = (1 + 2r)/3. \) Note that \( \det \mathcal{M}^2 = m_\pi^2(2r - 1)/z^2, \) so that \( \lambda_+ = (2r - 1)/z^2. \) The eigenvalues are plotted in the figure, with the solid line denoting \( \lambda_+, \) the dashed line denoting \( \lambda_-, \) and the dot-dashed line denoting the asymptotic value \( \lambda_+ \to (4r - 1)/3 = 17.07 \) as \( z \to \infty. \)
For $z \to 0$ the higher eigenvalue behaves as $\lambda_+ \to \gamma/z^2 = (2r + 1)/(3z^2) \to \infty$, while the lower eigenvalue behaves as $\det \mathcal{M}^2/\lambda_+ = 3(2r - 1)/(2r + 1) < 3$. From the figure one sees that $\lambda_-$ is monotonically decreasing with $z$, so $\mathcal{M}^2$ always has one eigenvalue below $3m\pi^2$.

(c) Find the relation between the lower eigenvalue and $F_0$ in the limit $F_0 \to \infty$. What is the implied mass for $F_0 = 10^{12}$ GeV?

**Answer:** In the limit $z \to \infty$, the large eigenvalue $\lambda_+$ approaches $\mathcal{M}_{88}/m_\pi^2 = (4r - 1)/3$, so using the fact that the product $\lambda_+\lambda_-$ of the eigenvalues is $\det \mathcal{M}^2/m_\pi^2$, we find $\lambda_- \to 3(2r - 1)/[(4r - 1)z^2]$. For $F_\pi = 93$ MeV and $F_0 = 10^{12}$ GeV, one finds an eigenvalue corresponding to a mass $m \simeq 1.5 \times 10^{-5}$ eV.

(d) Compare the higher eigenvalue in part (c) with the masses of the $\eta$ and $\eta'$, which you may find in the Particle Data Group listings,


To which mass is the higher eigenvalue closer?

**Answer:** The mass of $\eta(548)$ is about four times $m_\pi$, while the mass of $\eta'(958)$ is about seven times $m_\pi$. Hence in the limit $z \to \infty$, the higher eigenvalue $\lambda_+$ is closer to $(m_\eta/m_\pi)^2 \simeq 16$. 

2