DARK MATTER

Physics 472 - Winter Quarter, 2010 - University of Chicago

PROBLEM DUE TUESDAY, MARCH 9, AND ANSWERS

This exercise concerns the "U(1) problem" noted by S. Weinberg, Phys. Rev. D **11**, 3583 (1975). Current algebra is employed to parametrize the pseudoscalar meson masses as follows in the absence of an axial U(1) anomaly:

$$m_\pi^2 = c\left(\epsilon_0 + \frac{\epsilon_8}{\sqrt{2}}\right) \ , \quad m_K^2 = c\left(\epsilon_0 - \frac{\epsilon_8}{2\sqrt{2}}\right) \ , \tag{1}$$

and the physical η and η' are the eigenvalues of the 2×2 matrix

$$\mathcal{M}^2 = \begin{bmatrix} \mathcal{M}_{88}^2 & \mathcal{M}_{80}^2 \\ \mathcal{M}_{08}^2 & \mathcal{M}_{00}^2 \end{bmatrix} \ , \tag{2}$$

where

$$\mathcal{M}_{88}^2 \equiv c\left(\epsilon_0 - \frac{\epsilon_8}{\sqrt{2}}\right) \ , \quad \mathcal{M}_{80}^2 = \mathcal{M}_{08}^2 \equiv \frac{c}{z}\epsilon_8 \ , \quad \mathcal{M}_{00}^2 \equiv \frac{c}{z^2}\epsilon_0 \ . \tag{3}$$

Here $z = F_0/F_\pi$ is the ratio of the SU(3)-singlet decay constant F_0 to the pion decay constant $F_\pi = 93$ MeV.

(a) Express the matrix $\mathcal{M}^2$ in terms of m_π^2, m_K^2, and z.

Answer: Algebraic combinations of Eqs. (1) and (3) lead to

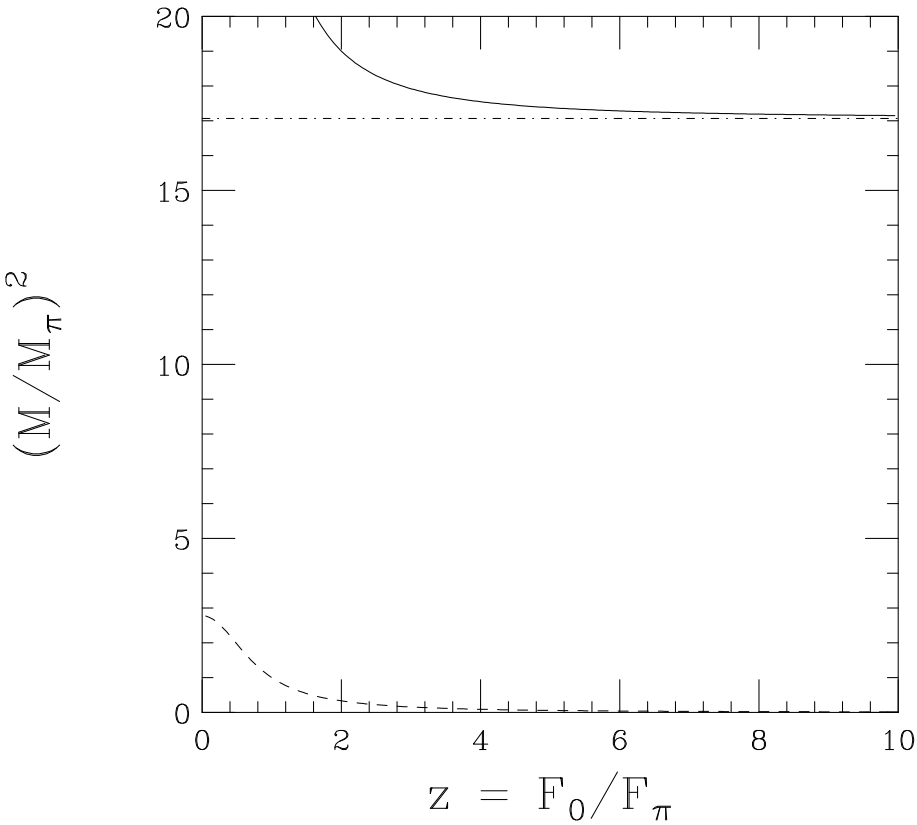
$$\mathcal{M}^2 = \begin{bmatrix} \frac{1}{3}\left(4m_K^2 - m_\pi^2\right) & -\frac{2\sqrt{2}}{3z}\left(m_K^2 - m_\pi^2\right) \\ -\frac{2\sqrt{2}}{3z}\left(m_K^2 - m_\pi^2\right) & \frac{1}{3z^2}\left(2m_K^2 + m_\pi^2\right) \end{bmatrix} \ . \tag{4}$$

(b) Show (either numerically or analytically) that the matrix always has at least one eigenvalue less than $3m_\pi^2$.

Answer: Writing $\mathcal{M}_{88} = \alpha m_\pi^2$, $\mathcal{M}_{08} = \mathcal{M}_{80} = \beta m_\pi^2/z$, $\mathcal{M}_{00} = \gamma m_\pi^2/z^2$, and $r = m_K^2/m_\pi^2 = 13.05$, the eigenvalues in units of m_π^2 are

$$\lambda_\pm = \frac{1}{2}\left(\alpha + \frac{\gamma}{z^2}\right) \pm \sqrt{\frac{1}{4}\left(\alpha - \frac{\gamma}{z^2}\right)^2 + \frac{\beta^2}{z^2}} \ , \tag{5}$$

where $\alpha = (4r-1)3$, $\beta = -2\sqrt{2}(r-1)/3$, and $\gamma = (1+2r)/3$. Note that $\det\mathcal{M}^2 = m_\pi^2(2r-1)/z^2$, so that $\lambda_+\lambda_- = (2r-1)/z^2$. The eigenvalues are plotted in the figure, with the solid line denoting λ_+, the dashed line denoting λ_-, and the dot-dashed line denoting the asymptotic value $\lambda_+ \to (4r-1)/3 = 17.07$ as $z \to \infty$.



For $z \to 0$ the higher eigenvalue behaves as $\lambda_+ \to \gamma/z^2 = (2r+1)/(3z^2) \to \infty$, while the lower eigenvalue behaves as $\det\mathcal{M}^2/\lambda_+ = 3(2r-1)/(2r+1) < 3$. From the figure one sees that λ_- is monotonically decreasing with z, so $\mathcal{M}^2$ always has one eigenvalue below $3m\pi^2$.

(c) Find the relation between the lower eigenvalue and F_0 in the limit $F_0 \to \infty$. What is the implied mass for $F_0 = 10^{12}$ GeV?

Answer: In the limit $z \to \infty$, the large eigenvalue λ_+ approaches $\mathcal{M}_{88}/m_\pi^2 = (4r-1)/3$, so using the fact that the product $\lambda_+\lambda_-$ of the eigenvalues is $\det\mathcal{M}^2/m_\pi^2$, we find $\lambda_- \to 3(2r-1)/[(4r-1)z^2]$. For $F_\pi = 93$ MeV and $F_0 = 10^{12}$ GeV, one finds an eigenvalue corresponding to a mass $m \simeq 1.5 \times 10^{-5}$ eV.

(d) Compare the higher eigenvalue in part (c) with the masses of the η and η', which you may find in the Particle Data Group listings,

`http://pdg.lbl.gov/2009/tables/rpp2009-tab-mesons-light.pdf`

To which mass is the higher eigenvalue closer?

Answer: The mass of $\eta(548)$ is about four times m_π, while the mass of $\eta'(958)$ is about seven times m_π. Hence in the limit $z \to \infty$, the higher eigenvalue λ_+ is closer to $(m_\eta/m_\pi)^2 \simeq 16$.