DARK MATTER

Physics 472 - Winter Quarter, 2010 - University of Chicago

PROBLEMS DUE TUESDAY, FEBRUARY 16 – ANSWERS

Problem 1: Consider a dark matter particle of mass M_D and velocity v_D scattering elastically on a target of mass M_T stationary in the laboratory system. Consider all motion to be non-relativistic.

(a) Calculate the velocity v^* of the center-of-mass (c.m.s.) system.

Answer: We use non-relativistic kinematics throughout. In the c.m.s, the momentum p_D^* of the dark matter particle is $p_D^* = M_D(v_D - v^*)$, equal and opposite to the target momentum $p_T^* = M_T(-v^*) = -p_D^*$. Solving f)or v^*, we have $v^* = M_D v_D/(M_D + M_T) = p_D/(M_D + M_T)$.

(b) Let the c.m.s. scattering angle be θ^*. Calculate the final longitudinal and transverse momenta in the lab system of the dark matter particle and the target, in terms of the target and projectile masses, v_D, and θ^*.

Answer: In the c.m.s, the longitudinal and transverse final 3-momenta of the dark matter particle are ($p^* \equiv p_D^* = -p_T^*$)

$$p'^*_{Dz} = p^* \cos\theta = \frac{p_D M_T}{M_D + M_T} \cos\theta \ , \quad p'^*_{D\perp} = p^* \sin\theta = \frac{p_D M_T}{M_D + M_T} \sin\theta \ . \tag{1}$$

Transforming to the lab frame, they become

$$p'_{Dz} = M_D v^* + p'^*_{Dz} = \frac{p_D}{M_D + M_T} \left(M_D + M_T \cos\theta\right) \ , \quad p'_{D\perp} = p'^*_{D\perp} \ . \tag{2}$$

The corresponding expressions for the target particle are

$$p'_{Tz} = \frac{p_D M_T}{M_D + M_T}(1 - \cos\theta) \ , \quad p'_{T\perp} = -\frac{p_D M_T}{M_D + M_T} \sin\theta \ . \tag{3}$$

For $\theta = 0$, $p'_{DZ} = p_D$ and $p'_{Tz} = 0$. For $\theta = \pi$, one has $p'_{Dz} = (M_D - M_T)/(M_D + M_T)p_D$ and $p'_{Tz} = (2M_T p_D)(M_D + M_T)$.

(c) Calculate the kinetic energies E'_D and E'_T of the final dark matter and target particles in terms of the kinetic energy E_D of the initial dark matter particle and the c.m.s. scattering angle θ^*. For each value of θ^*, show that E'_T attains a maximum for some ratio of M_D/M_T and calculate this ratio. Assuming isotropic scattering in the c.m.s., calculate the average values $\langle E'_D \rangle$ and $\langle E'_T \rangle$.

Answer: Knowing the longitudinal and transverse lab momenta of the scattered dark matter particle and the target, we find their final kinetic energies

$$E'_D = \frac{p'^2_{Dz} + p'^2_{D\perp}}{2M_D} = E_D \left[1 - \frac{2M_D M_T}{(M_D + M_T)^2}(1 - \cos\theta)\right] \ , \tag{4}$$

$$E'_T = E_D \left[\frac{2M_D M_T}{(M_D + M_T)^2}(1 - \cos\theta)\right] = E_D - E'_D \ . \tag{5}$$

The maximum value of E'_T for a fixed θ occurs for the maximum of $(2M_D M_T)/(M_D + M_T)^2$, which is 1/2 when $M_D/M_T = 1$. For isotropic scattering in the c.m.s., the average value of $\cos\theta$ is 0, so

$$\langle E'_D \rangle = E_D \left[\frac{M_D - M_T}{M_D + M_T} \right]^2 \ , \quad \langle E'_T \rangle = E_D \left[\frac{2M_D M_T}{(M_D + M_T)^2} \right] \ . \tag{6}$$

(d) In the lecture on January 28, it was stated that recoil energies in the keV range could be achieved using 24 keV neutrons on heavy nuclei. Assuming isotropic scattering in the c.m.s., calculate the shape and endpoint of the recoil spectra for targets of Ne (A = 20), Ar (A = 40), and Xe (A = 131).

Answer: The spectrum of recoil kinetic energies is flat from zero to a maximum value of

$$\left[\frac{E'_T}{E_D} \right]_{\max} = \frac{4M_D M_T}{(M_D + M_T)^2} = \frac{4A}{(A+1)^2} \ , \tag{7}$$

where A is the atomic number of the target and we have used the approximation that the atomic number of the neutron is 1. This quantity is 0.180 for Ne ($A = 20.18$), 0.095 for Ar ($A = 39.95$), and 0.030 for Xe ($A = 131.30$), corresponding to respective target recoil energies of 4.32, 2.29, and 0.72 KeV for incident 24 keV neutrons.

Problem 2: A simplified model of the halo of the Milky Way may be made by assuming that the average circulation velocity of stars in circular orbits around the center of the Galaxy is constant at the local velocity of the Sun, which is $v_\oplus = 220$ km/s at the Sun's distance $r_0 = 8.5$ kpc from the center of the Galaxy, *independent of the radius r of the orbit*. Assume this behavior extends out to a radius $r_{\rm outer}$, beyond which the density of the Galaxy drops abruptly to zero.

(a) Derive a relation between the velocity $v_{\rm esc}$ that a test particle initially at r_0 needs to escape the Galaxy, the Sun's circulation velocity $v_\oplus$, and the ratio $r_{\rm outer}/r_0$.

Answer: Gauss' Law says that the force on a body of mass m at a radius r from the center of a spherically symmetric distribution of matter is given in terms of the mass $M(r)$ within r: $F(r) = -GM(r)m/r^2$. If this force balances the centripetal acceleration v^2/r and v is independent of r, the mass $M(r)$ must grow linearly with r, so the density may be parametrized as

$$\rho(r) = \rho_0 (r_0/r)^2 \ (r \le r_{\rm outer}) \ , \quad \rho(r) = 0 \ (r > r_{\rm outer}) \ . \tag{8}$$

One can then write

$$F(r) = -\frac{4\pi G \rho_0 \ r_0^2 \ m}{r} = -\frac{dV(r)}{dr} \ , \tag{9}$$

We need to determine the depth of the potential $V(r)$ at our radius $r_0 = 8.5$ kpc in order to determine the Galactic escape velocity. Integrating the above equation, we find

$$V(r_{\rm outer}) - V(r_0) = -\int_{r_0}^{r_{\rm outer}} dr F(r) = 4\pi G \rho_0 \ r_0^2 \ m \ln \frac{r_{\rm outer}}{r_0} \ . \tag{10}$$

For $r > r_{\rm outer}$,

$$V(r) - V(r_{\rm outer}) = \int_{r_{\rm outer}}^{r} dr \frac{4\pi G\rho_0\ r_0^2\ r_{\rm outer}}{r^2} \Rightarrow 4\pi G\rho_0\ r_0^2\ m \quad (r \to \infty)\ . \tag{11}$$

Then we can express the kinetic energy for escape, with escape velocity v_e, as

$$\frac{1}{2} m v_e^2 = -V(r_0) = 4\pi G\rho_0\ r_0^2\ m \left[\ln \frac{r_{\rm outer}}{r_0} + 1\right]\ . \tag{12}$$

Now, the circulation velocity $v_\oplus$ at r_0 is given by the balance between gravitational and centripetal acceleration,

$$\frac{4\pi G\rho_0\ r_0^3\ m}{r_0^2} = \frac{m v_\oplus^2}{r_0} \Rightarrow 4\pi G\rho_0\ r_0^2 = v_\oplus^2\ . \tag{13}$$

We thus have

$$v_e^2 = 2 v_\oplus^2 \left[\ln \frac{r_{\rm outer}}{r_0} + 1\right]\ . \tag{14}$$

(b) Assuming $r_{\rm outer} = 100$ kpc, what is $v_{\rm esc}$?

Answer: We have

$$v_e^2 = 2(220\ {\rm km/s})^2 \left[\ln \frac{100}{8.5} + 1\right] \Rightarrow v_e^2 = 579\ {\rm km/s}\ . \tag{15}$$

(c) In this model what is the local halo matter density at $r = r_0$? Express your answer in units of GeV/cm^3.

Answer: Using Eq. (13), we find

$$\rho_0 = \frac{1}{4\pi G} \left(\frac{v_\oplus}{r_0}\right)^2 = 0.50\ m_p/{\rm cm}^3 = 0.47\ {\rm GeV/cm}^3\ , \tag{16}$$

This is fairly close to the accepted value of 0.4 GeV/cm^3. Here we have used $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $v_\oplus = 220$ km/s, $r_0 = 8.5$ kpc $= (8500)(3.086 \times 10^{16}$ m), $m_p = 1.673 \times 10^{-27}$ kg.

(d) What is the total calculated mass of the Milky Way (in units of Solar mass $M_\odot$)?

Answer: The total mass is $M = 4\pi\rho_0\ r_0^2\ r_{\rm outer} = (v_\oplus^2/G) r_{\rm outer}$. We can evaluate this by comparing it with $GM_\odot = v_{\rm orb}^2 \cdot (1\ {\rm AU})$, where $v_{\rm orb} \simeq 30$ km/s is the orbital velocity of the Earth around the Sun, obtaining

$$\frac{M}{M_\odot} = \left(\frac{v_\oplus}{v_{\rm orb}}\right)^2 \frac{100\ {\rm kpc}}{1\ {\rm AU}} = \left(\frac{220}{30}\right)^2 \cdot 2.06 \times 10^{10} = 1.1 \times 10^{12}\ . \tag{17}$$