DARK MATTER

Physics 472 - Winter Quarter, 2010 - University of Chicago

PROBLEMS DUE TUESDAY, JANUARY 19 - ANSWERS

Problem 1: The original evidence for dark matter was published by Fritz Zwicky in Helvetica Physica Acta **6**, 110 (1933). An account of his method in English may be found in Astrophysical Journal **86**, 217 (1937). You are asked to update his calculation of the mass of the Coma Cluster. An exercise guiding you through this calculation may be found in

`http://spiff.rit.edu/classes/phys440/lectures/gal_clus/gal_clus.html`

It may be helpful to refer to an earlier exercise

`http://spiff.rit.edu/classes/phys440/lectures/glob_clus/glob_clus.html`

in which the mass of a globular cluster is calculated.

Answer: You are asked to download a table of radial velocities of galaxies:

`http://cdsweb.u-strasbg.fr/viz-bin/VizieR?-source=J/A+AS/111/265`

and will obtain a file looking like the one on the next page. The columns are as follows: (1) Galaxy serial number in set; (2) a galaxy identification number (including four of the brightest listed in the NGC catalog); (3) and (4) distance from the center of the cluster in arcseconds, west and north respectively; (5) apparent magnitude; (6) ? (irrelevant); (7) radial velocity in km/s; (8) and (9) ? (irrelevant). You are first asked to eliminate the outliers, with radial velocities far away from the average; these are probably not in the Cluster. The ones I dropped were 7, 12, 15, 16, 19, 26, 27, and 30, leaving a sample of 26.

Using the coordinates (3) and (4), one must calculate the distance to the center of the cluster using the Pythagorean Theorem: $d = (W^2 + N^2)^{1/2}$. The mean distance of the 26 galaxies is found to be $\bar{d} = 2486.9$ arcsec or $0.6908^\circ = 1.206 \times 10^{-2}$ radians. If the Coma Cluster is 100 Mpc away (the number quoted by Michael Richmond), this corresponds to a mean distance of 1.206 Mpc from the center of the cluster.

The *dispersion* σ_r of the radial velocities v_r is an indication of thermal behavior. Using the table of the 26 radial velocities (measured spectroscopically using Doppler shifts), one calculates a mean value $\bar{v}_r = 6971.8$ km/s and a squared dispersion $\sigma_r^2 = \overline{v_r^2} - \bar{v}_r^2 = 1.138 \times 10^{16}$ m^2 s^{-2}. (A program which calculates these averages and the data it uses may be found on p. 3.) One should multiply σ_r^2 by 3 to get the total squared dispersion: $\sigma^2 = 3\sigma_r^2 = 3.413 \times 10^{16}$ m^2 s^{-2}. (For comparison, Zwicky had 5×10^{15} m^2 s^{-2}, using an estimate of 13.7 Mpc for the distance to the Coma Cluster. As mentioned in class, distance scales were seriously underestimated until the mid-1950s.)

```
# data from Velocities in Coma cluster (Biviano+ 1995)
# http://cdsweb.u-strasbg.fr/viz-bin/VizieR?-source=J/A+AS/111/265
#                                                               RV
#                          W      N                          km/sec
    1     4469             1310  -1414  17.69   1.88    7452   22 2
    2     4479             1322  -1982  17.51   1.83    5749   16 2
    3     4522             1380  -1728  15.83   1.84    7606   12 2
    4     4535             1393  -1857  17.90   1.95    7653   13 2
    5     4579             1451  -1396  16.72           4915   53 2
    6     4578             1451  -2113  18.04   1.80    5186   29 2
    7     4592             1470  -1272  16.38   2.09   20220      4
    8     4597             1478  -1723  16.37   1.91    4915   25 2
    9     4630             1525  -1721  18.97   1.92    7335   17 2
   10     4692             1594  -1944  17.37   1.74    8318   17 2
   11     4714             1624  -1410  17.54   1.75    7226   18 2
   12     4749             1665  -1540  19.17   1.72   49310   71 1 3
   13     4792 NGC4842B    1721  -1743  16.30   1.91    7173   32 1 1
   14     4794 NGC4842A    1724  -1712  15.26   2.02    7304   28 1 1
   15                      1740  -1720                  -154   40 1 1
   16     4825             1758  -1700  19.34   1.63   19865   15 1 2 E
   17     4829 NGC4840     1762  -1291  14.86   1.98    6055   20 2
   18     4852             1791  -1532  18.36   1.97    7694   45 1 1
   19     4858             1797  -1799  18.78   2.10   38272   60 1 1
   20     4907             1855  -1522  15.93   2.20    5504   36 1 1
   21     4918             1867  -2030  16.03           4811   55 2
   22     4928 NGC4839     1877  -1694  13.51           7442  100 3
   23     4937             1887  -1506  18.35   1.82    5709  119 1 3
   24     4943             1896  -1712  15.88   1.83    8203   27 1 1
   25     4956             1913  -1778  18.52   1.73    6819   48 1 1
   26  *  5323             1950  -1810  19.72   2.19    -101   37 1 1
   27  *  5348             1980  -1795  15.41   2.19    -354  139 1 2
   28     5038             2060  -2028  16.14   1.82    6205   15 2
   29     5051             2076  -1806  15.46   2.07    7323   35 1 1
   30  *  5439             2085  -1520  18.75   1.68    -202   40 1 1
   31     5102             2141  -1591  17.50   1.92    8122   93 1 2
   32     5136             2182  -2146  16.62   2.00    7012   10 2
   33     5284             2434  -1509  17.98   1.75    7545   37 2
   34     5296             2454  -1872  18.90   1.83    7310   21 2
```

```
      program rvdist
      implicit real*8(a-h,o-z)
      open(unit=9,file='rvdist.dat',status='unknown')
      open(unit=10,file='rvout.dat',status='unknown')
      v = 0.
      vs = 0.
      d = 0.
      do i=1,26
      read (9,*) dist, rv
      d = d + dist
      v = v + rv
      vs = vs + rv*rv
      end do
      d = d/26.
      v = v/26.
      vs = vs/26.
      sig = sqrt(vs - v*v)
      write (6,703) d, v, sig
      write (10,703) d, v, sig
 703  format('Avg. dist. ',F8.2,' arcsec; avg. v ',F8.2,' km/s;
     + sigma = ',F8.2,' km/s')
      close(unit=9)
      close(unit=10)
      stop
      end
```

```
File rvdist.dat (needs to be in two columns to run):
    d       v         d       v         d       v
   1928.   7452.     2151.   7226.     2555.   8203.
   2382.   5749.     2449.   7173.     2612.   6819.
   2211.   7606.     2430.   7304.     2891.   6205.
   2321.   7653.     2184.   6055.     2752.   7323.
   2014.   4915.     2357.   7694.     2667.   8122.
   2563.   5186.     2399.   5504.     3060.   7012.
   2270.   4915.     2758.   4811.     2864.   7545.
   2299.   7335.     2528.   7442.     3087.   7310
   2514.   8318.     2414.   5709.
```

```
File rvout.dat:
Avg. dist. 2486.92 arcsec; avg. v 6791.77 km/s; sigma =  1066.60 km/s
```

We may now use the virial theorem, which says that twice the expectation value of the (thermal) kinetic energy of a test particle is the average of its potential energy. For a particle a distance r from a mass M (which may be distributed in any spherically symmetric manner within the radius r) we then have $\sigma^2 = GM/r$. One evaluates M most easily by comparison with the orbital velocity of the Earth, $v_\oplus = 29.79$ km/s, a distance of 1 AU (Astronomical unit $= 1.496 \times 10^8$ km) from the Sun with mass $M_\odot$:

$$\frac{\sigma^2}{v_\oplus^2} = \frac{M}{M_\odot} \frac{1 \text{ AU}}{1.206 \text{ Mpc}} = 3.847 \times 10^3 \ . \tag{1}$$

We also use 1.206 Mpc $= (1.206) \cdot 2.063 \times 10^{11} = 2.488 \times 10^{11}$ so that

$$\frac{M}{M_\odot} = (3.847 \times 10^3)(2.488 \times 10^{11}) = 9.57 \times 10^{14} \ . \tag{2}$$

2. The Friedman equation for the Hubble constant H in terms of the energy density (assuming a flat Universe, with zero curvature) may be trivially integrated to give the time-dependence of the scale factor $a(t)$ if the scale-dependence of the energy density is known. For example, the matter energy density scales as the inverse cube of $a(t)$, while radiation energy density scales as the inverse fourth power. Some of these integrals were done in class.

(a) Use the second Friedmann equation [given in class, or see, e.g., Carroll, Eq. (8.68)], which involves both the energy density and the pressure, to relate the ratio w of pressure and energy density to the scale-dependence parameter of the energy density. (A similar relation may be obtained directly using the conservation of the energy-momentum tensor.)

(b) Suppose you have a Universe consisting of a fraction f of dark energy and $1 - f$ of matter, with total energy density equal to the critical density. Integrate the (first) Friedmann equation to relate the scale factor $a(t)$ to the time before the present (in units of the inverse Hubble time). Either numerical or analytic integration will be acceptable. Compare your results for $f = 0$, 0.72, and 1.

Answer: (a) the Friedmann equations (for a flat Universe) are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \ , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}\rho(1 + 3w) \tag{3}$$

Let $\rho = \rho^0 a^{-\alpha}$. The first Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-\alpha} = H_0^2 \Omega_0 a^{-\alpha} \ , \tag{4}$$

where $H_0^2 = 8\pi G \rho_{\text{crit}}/3$ and $\Omega_0 = \rho_0/\rho_{\text{crit}}$. This may be solved to give

$$a = \left(\frac{\alpha}{2} H_0 \sqrt{\Omega_0} t\right)^{\alpha/2} \ (\alpha \neq 0) \ ; \quad a = \exp(H_0 \sqrt{\Omega_0} t) \ (\alpha = 0) \ . \tag{5}$$

Then (for $\alpha \neq 0$)

$$\frac{\ddot{a}}{a} = \frac{2}{\alpha}\left(\frac{2}{\alpha} - 1\right)\frac{1}{t^2} = -\frac{1}{2}H_0\sqrt{\Omega_0}(1+3w) \ . \tag{6}$$

Demanding consistency with $(\dot{a}/a)^2 = (2/\alpha)^2 t^{-2}$, we find

$$\frac{1+3w}{2} = \left(1 - \frac{2}{\alpha}\right)/\frac{2}{\alpha} = \frac{\alpha}{2} - 1 \ \Rightarrow \ \alpha = 3(1+w) \ . \tag{7}$$

For $\alpha = 0$ consistency between the two Friedmann equations leads more directly to $w = -1$.

2. For a matter-dominated Universe with $\Omega_m^0 = 1$, so that $\rho = \rho_{\text{crit}}/a^3$, the first Friedmann equation may be expressed in terms of the lookback time $\tilde{t}$ with the condition $a(0) = 1$ to give $a = \left(1 - \frac{3}{2}H_0\tilde{t}\right)^{2/3}$. For a Universe where $\rho_{\text{DE}} = \rho_{\text{crit}}$, the solution is $a = \exp(-H_0\tilde{t})$. For a Universe with a fraction $f = \Omega_0^m$ of matter and $1 - f = \Omega_{DE}$ of dark energy, the first Friedmann equation becomes

$$\frac{da}{d\tilde{t}} = -H_0\left(\frac{f}{a} + (1-f)a^2\right)^{1/2} \ . \tag{8}$$

The solution of this is

$$H_0\tilde{t} = \int_a^1 \frac{\sqrt{a'}da'}{\sqrt{f + (1-f)a'^3}} \ , \tag{9}$$

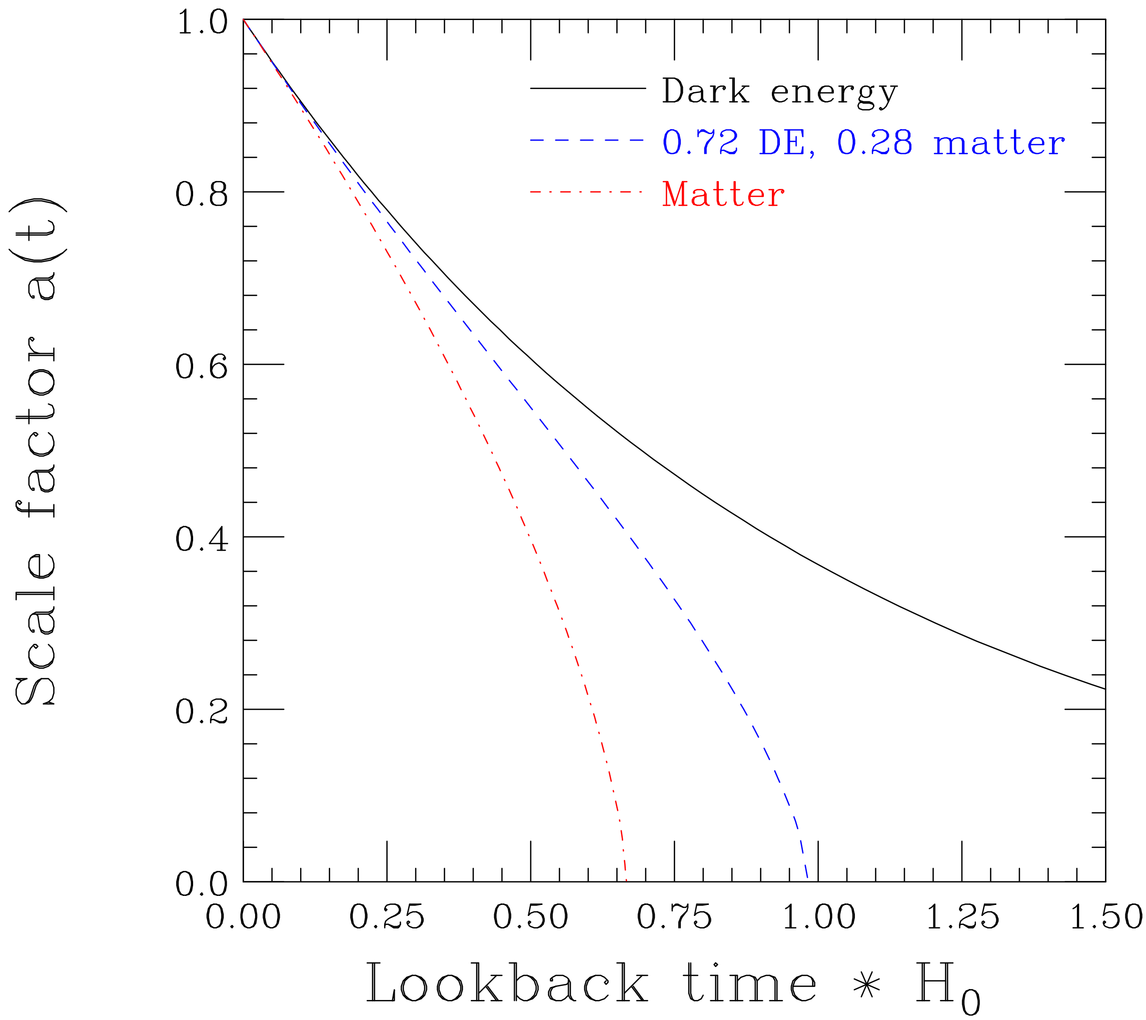
which may be expressed as an elementary integral with the substitution $\alpha \equiv a'^{3/2}$ so that

$$\begin{aligned} \frac{3}{2}H_0\tilde{t} &= \int_{a^{3/2}}^1 \frac{d\alpha}{f + (1-f)\alpha^2} \\ &= \frac{1}{\sqrt{1-f}}\left[\sinh^{-1}\sqrt{\frac{1-f}{f}} - \sinh^{-1}\left(\sqrt{\frac{1-f}{f}}a^{3/2}\right)\right] \ . \end{aligned} \tag{10}$$

An equivalent function for $\sinh^{-1}$ is $\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$. The limits of Eq. (10) for $f \to 0, \ 1$ are:

$$f \to 0 \ : \ a(t) \to \exp(-H_0\tilde{t}) \ : \ f \to 1 \ : \ a(t) \to \left(1 - \frac{2}{3}H_0\tilde{t}\right)^{2/3} \ . \tag{11}$$

The three cases of $f = 0$ (pure dark energy, upper curve), $f = 0.28$ (mixture, middle curve), and $f = 1$ (pure matter, lower curve) are plotted on the next page. The age of the Universe is $(2/3)H_0^{-1}$ for a matter-dominated Universe, just about H_0^{-1} for $f = 0.28$, and infinite for a Universe dominated by dark energy.



Dark energy
0.72 DE, 0.28 matter
Matter
Scale factor a(t)
Lookback time * H0
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50