QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEM DUE TUESDAY, FEBRUARY 28

Problem in text	**Subject**
12–1	Beta functions in mixed ϕ^4–Yukawa theory

Useful results to recall include those on pp. 327-329 of the text. In addition you will need the contribution to the $-i\delta_\lambda$ counterterm from the fermion loop, and the δ_Z contribution to the fermion Green's function from the loop with one internal boson and one internal fermion.

Solutions — Problem Set 7

David McKeen – February 28, 2005

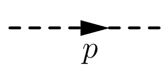
Problem 12–1 We begin with the bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \bar{\psi}(i\not{\partial} - m_0)\psi - ig_0\bar{\psi}\gamma^5\psi\phi \quad .$$

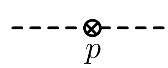
We rescale the fields: $\phi \to \sqrt{Z_\phi}\phi$, $\psi \to \sqrt{Z_\psi}\psi$. We define the counterterms $\delta_\phi = Z_\phi - 1$, $\delta_\psi = Z_\psi - 1$, $\delta_\lambda = \lambda_0 Z_\phi^2 - \lambda$, $\delta_g = g_0\sqrt{Z_\phi}Z_\psi - g$, $\delta_M = Z_\phi M_0^2$, and $\delta_m = Z_\psi m$. The last two reflect the fact that we have a massless theory; we assume that the *observed* particles have $M = m = 0$. We then obtain the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\not{\partial})\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{1}{2}\delta_M\phi^2 - \frac{\delta_\lambda}{4!}\phi^4 + \bar{\psi}(i\delta_\psi\not{\partial} - \delta_m)\psi - i\delta_g\bar{\psi}\gamma^5\psi\phi$$

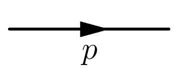
with the following Feynman rules:



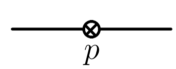
$= \dfrac{i}{p^2}$



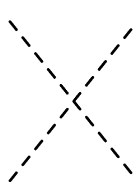
$= i(p^2\delta_\phi - \delta_M)$

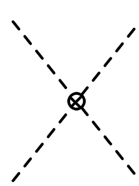


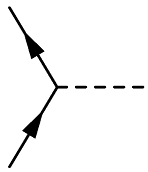
$= \dfrac{i\not{p}}{p^2}$



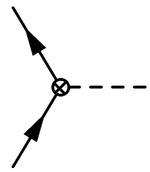
$= i(\not{p}\delta_\psi - \delta_m)$


$= -i\lambda$


$= -i\delta_\lambda$



$= g\gamma^5$



$= \delta_g \gamma^5$

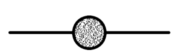
We impose renormalization conditions. We define the sum of 1PI diagrams involving the scalar field:

$= -iM^2(p^2)$

We define the theory at spacelike momentum P^2:

$$\begin{aligned} M^2(p^2 = -P^2) &= 0 \\ \frac{dM^2}{dp^2}(p^2 = -P^2) &= 0 \quad . \end{aligned}$$

This keeps the scalar field massless at this scale. (In fact it will be massless at all scales. One can see this in Peskin and Schroeder Section 12.4 – the mass “beta function” is proportional to the renormalized mass.) We similarly define the sum of all 1PI diagrams involving the fermion field:

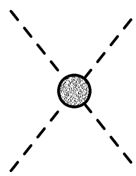


$= -i\Sigma(\not p)$

We impose the conditions

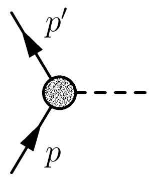
$$\begin{aligned} \Sigma(\not p = P) &= 0 \\ \frac{d\Sigma}{d\not p}(\not p = P) &= 0 \end{aligned}$$

to keep the field massless. We then set the values of the vertices:



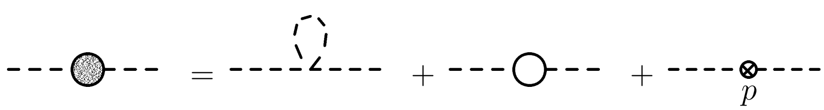
$$= -i\lambda \quad \text{at } s = -P^2,\ t = u = 0 \quad ;$$

and



$$= g\gamma^5 \quad \text{at } q = p' - p = P \quad .$$

We first calculate the correction to the scalar propagator (the symbol $\sim$ translates here to "diverges as" since we are only interested in the divergent parts of counterterms):



$$= i\mathcal{M}_1 + i\mathcal{M}_2 + i(p^2\delta_\phi - \delta_M) \quad .$$

We calculate the first diagram:

$$\begin{aligned} i\mathcal{M}_1 &= \lambda \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} \\ &= \lim_{\mu\to 0} \lambda \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2} \\ &= \lim_{\mu\to 0} \frac{-i\lambda}{4\pi}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{4\pi}{\mu^2}\right)^{1-\frac{d}{2}} \\ &\sim \lim_{\mu\to 0} \frac{+i\lambda\mu^2}{(4\pi)^2}\left(\frac{2}{\epsilon}\right) \quad . \end{aligned}$$

The second diagram is

$$
\begin{aligned}
i\mathcal{M}_2 &= g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}[\gamma^5(\not{k}+\not{p})\gamma^5\not{k}]}{k^2(k+p)^2} \\
&= -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}[(\not{k}+\not{p})\not{k}]}{k^2(k+p)^2} \\
&= -4g^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k+p)\cdot k}{k^2(k+p)^2} \quad .
\end{aligned}
$$

We introduce a Feynman parameter and the denominator becomes

$$
x(k+p)^2 + (1-x)k^2 = \ell^2 - \Delta
$$

with $\ell = k + xp$ and $\Delta = x(x-1)p^2$. The numerator becomes

$$
(k+p)\cdot k = [\ell - (x-1)p]\cdot[\ell - xp] \to \ell^2 + \Delta
$$

when terms linear in ℓ are ignored. Then

$$
i\mathcal{M}_2 = -4g^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2+\Delta}{(\ell^2-\Delta)^2} \quad .
$$

Using Eqs. (A.) and (A.) along with the fact that $x\Gamma(x) = \Gamma(x+1)$ we find that

$$
\begin{aligned}
i\mathcal{M}_2 &= \frac{4ig^2}{(4\pi)^2}\Gamma\left(1-\frac{d}{2}\right)\int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^{2-\frac{d}{2}} \left[\frac{d}{2}\Delta - (1-\frac{d}{2})\Delta\right] \\
&\sim \frac{4ig^2}{(4\pi)^2}\left(-\frac{2}{\epsilon}\right)\int_0^1 dx\ 3\Delta \\
&= \frac{2ig^2p^2}{(4\pi)^2}\left(\frac{2}{\epsilon}\right) \quad .
\end{aligned}
$$

Collecting these results we find that

$$
-iM^2(p^2) \sim \lim_{\mu\to 0} \frac{+i\lambda\mu^2}{(4\pi)^2}\left(\frac{2}{\epsilon}\right) + \frac{2ig^2p^2}{(4\pi)^2}\left(\frac{2}{\epsilon}\right) + i(p^2\delta_\phi - \delta_M) \quad .
$$

To satisfy our renormalization condition we require

$$\delta_\phi \sim -\frac{2g^2}{(4\pi)^2}\left(\frac{2}{\epsilon}\right)$$

or

$$A_\phi = -\frac{2g^2}{(4\pi)^2}$$

and

$$\delta_M \sim \lim_{\mu\to 0} -\frac{\lambda\mu^2}{(4\pi)^2}\left(\frac{2}{\epsilon}\right) \quad .$$

The correction to the fermion propagator is

$$= i\mathcal{M}_3 + i(\not{p}\,\delta_\psi - \delta_m) \quad .$$

The first diagram on the right side is

$$\begin{aligned} i\mathcal{M}_3 &= -g^2 \int \frac{d^4k}{(2\pi)^4}\frac{\gamma^5\,\not{k}\gamma^5}{k^2(p-k)^2} \\ &= g^2 \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}}{k^2(p-k)^2} \\ &= g^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{\not{\ell} + x\,\not{p}}{(\ell^2-\Delta)^2} \end{aligned}$$

with $\ell = k - xp$ and $\Delta = x(x-1)p^2$. The term linear in ℓ drops out and we get

$$\begin{aligned} i\mathcal{M}_3 &= \frac{ig^2}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^{2-\frac{d}{2}} x\,\not{p} \\ &\sim \frac{ig^2\,\not{p}}{2(4\pi)^2}\left(\frac{2}{\epsilon}\right) \quad . \end{aligned}$$

This gives

$$-i\Sigma(\not{p}) \sim \frac{ig^2\,\not{p}}{2(4\pi)^2}\left(\frac{2}{\epsilon}\right) + i(\not{p}\delta_\psi - \delta_m) \quad .$$

This implies that

$$\delta_\psi \sim -\frac{g^2}{2(4\pi)^2}\left(\frac{2}{\epsilon}\right)$$

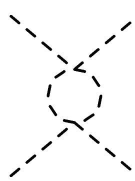
or

$$A_\psi = -\frac{g^2}{2(4\pi)^2}$$

and

$$\delta_m \sim 0 \quad .$$

We now calculate the corrections to the 4-boson interaction. We first consider the diagram with an internal boson loop. There are s-, t-, and u-channels. By crossing symmetry we can calculate one and simply make the change of Mandelstam variable to arrive at the others. We calculate the s-channel:



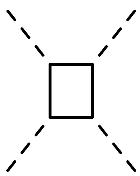
$$= iV(p^2)$$

$$= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k+p)^2}$$
$$= \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2}$$

with $\ell = k + xp$ and $\Delta = x(x-1)p^2$. We evaluate this integral to get

$$iV(p^2) = \frac{i\lambda^2}{2(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^{2-\frac{d}{2}}$$
$$\sim \frac{i\lambda^2}{2(4\pi)^2}\left(\frac{2}{\epsilon}\right) \quad .$$

We see that the divergent terms are independent of momentum and therefore the s-,t-, and u-channels will all give the same result. Our renormalization conditions simplify our calculation of the 4-point boson interaction with an internal fermion loop. As in the light-by-light scattering case there are 6 diagrams which contribute. However, we are only interested in the $s = -P^2$, $t = u = 0$ case; here all 4 external momenta are equal. We call this momentum p. Then all 6 diagrams give the same result. We calculate one of them:



$$= i\mathcal{M}$$

$$
\begin{aligned}
&= -g^4 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}[\gamma^5(\not{k} - \not{p})\gamma^5 \not{k} \gamma^5 (\not{k} + \not{p})\gamma^5 \not{k}]}{(k^2)^2(k-p)^2(k+p)^2} \\
&= -g^4 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}[(\not{k} - \not{p}) \not{k} (\not{k} + \not{p}) \not{k}]}{(k^2)^2(k-p)^2(k+p)^2} \\
&= -3!g^4 \int_0^1 dx_1 dx_2 dx_3 dx_4 \, \left(\sum_i x_i - 1 \right) \int \frac{d^4\ell}{(2\pi)^4} \frac{N}{D^4}
\end{aligned}
$$

where $D = \ell^2 - \Delta$ with $\ell = k - (x_3 - x_4)p$ and $\Delta = (x_3 - x_4)^2 p^2$. The leading behavior of N is $\mathrm{tr}(\not{\ell} \not{\ell} \not{\ell} \not{\ell}) \sim 4(\ell^2)^2$. Since x_1 and x_2 don't appear in the integrand we can easily integrate them to find

$$
\begin{aligned}
i\mathcal{M} &\sim -24g^4 \int_0^1 dx_3 \int_0^{1-x_3} dx_4 \, (1 - x_3 - x_4) \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^2)^2}{(\ell^2 - \Delta)^4} \\
&= -\frac{24ig^4}{(4\pi)^2} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx_3 \int_0^{1-x_3} dx_4 \, (1 - x_3 - x_4) \left(\frac{4\pi}{\Delta}\right)^{2-\frac{d}{2}} \\
&\sim -\frac{24ig^4}{(4\pi)^2} \left(\frac{2}{\epsilon}\right) \int_0^1 dx_3 \int_0^{1-x_3} dx_4 \, (1 - x_3 - x_4) \\
&= -\frac{4ig^4}{(4\pi)^2} \left(\frac{2}{\epsilon}\right) \quad .
\end{aligned}
$$

The total correction the 4-boson interaction is then

$$iV(-P^2) + 2iV(0) + 6i\mathcal{M} - i\delta_\lambda = 0 \quad .$$

This gives

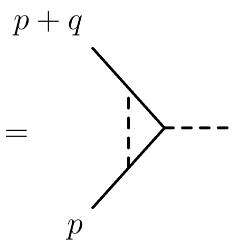
$$\delta_\lambda \sim \frac{1}{2(4\pi)^2}(3\lambda^2 - 48g^4)\left(\frac{2}{\epsilon}\right)$$

or

$$B_\lambda = -\frac{1}{2(4\pi)^2}(3\lambda^2 - 48g^4) \quad .$$

The lowest order correction to the Yukawa vertex is the diagram

$$\bar{u}(p+q)\delta\Gamma u(p) =$$



$$\begin{aligned}
\bar{u}(p+q)\delta\Gamma u(p) &= -ig^3\bar{u}(p+q)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^5(\not{k}+\not{q})\gamma^5\,\not{k}\gamma^5}{k^2(k+q)^2(p-k)^2}u(p) \\
&= -ig^3\bar{u}(p+q)\int \frac{d^4k}{(2\pi)^4}\frac{(\not{k}+\not{q})\gamma^5\,\not{k}}{k^2(k+q)^2(p-k)^2}u(p) \\
&= -2ig^3\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \\
&\quad \times \int \frac{d^4\ell}{(2\pi)^4}\frac{\bar{u}(p+q)(\not{\ell} + x_1\,\not{p} + (1-x_2)\,\not{q})\gamma^5(\not{\ell} + x\,\not{p} - x_2\,\not{q})u(p)}{(\ell^2-\Delta)^3}
\end{aligned}$$

with $\ell = k - x_1 p + x_2 q$ and $\Delta = (x_1 p - x_2 q)^2 - x_1 p^2 - x_2 q^2$. The divergent part of the integral can be written

$$
\begin{aligned}
\delta\Gamma &\sim -2ig^3 \int_0^1 dx\ (1-x) \int \frac{d^4\ell}{(2\pi)^4} \frac{\not{\ell}\gamma^5 \not{\ell}}{(\ell^2-\Delta)^3} \\
&= 2ig^3\gamma^5 \int_0^1 dx \int_0^{1-x_1} dx_2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta)^3} \\
&= -\frac{2g^3\gamma^5}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx \int_0^{1-x_1} dx_2 \left(\frac{4\pi}{\Delta}\right)^{2-\frac{d}{2}} \\
&\sim -\frac{g^3\gamma^5}{(4\pi)^2}\left(\frac{2}{\epsilon}\right) \quad .
\end{aligned}
$$

The total correction to the Yukawa vertex is

$$
\delta\Gamma + \delta_g\gamma^5 = 0 \quad .
$$

This implies

$$
\delta_g \sim \frac{g^3}{(4\pi)^2}\left(\frac{2}{\epsilon}\right)
$$

or

$$
B_g = -\frac{g^3}{(4\pi)^2} \quad .
$$

The Callan-Symanzik equations don't depend on mass counterterms since the theory is at a fixed point when it is massless (which can be seen in Peskin and Schroeder's discussion of the running of masses). Then we can use the simple formulae from Peskin and Schroeder to find the beta functions in terms of the coefficients of divergent logarithms (or $2/\epsilon$ to one loop order) of the counterterms. We can now read off the beta functions for the theory:

$$
\begin{aligned}
\beta_\lambda(\lambda, g) &= -2B_\lambda - 4\lambda A_\phi \\
&= \frac{1}{(4\pi)^2}(3\lambda^2 + 8\lambda g^2 - 48g^4)
\end{aligned}
$$

and

$$\beta_g(\lambda, g) = -2B_g - 2gA_\psi - gA_\phi$$
$$= \frac{5g^3}{(4\pi)^2} \quad .$$

We now investigate the flow of the coupling constants as we change the renormalization scale. We make the change of variables $\log(p/P) = (4\pi)^2 t$ and $g^2 = \nu$. We can then write the beta functions as (primes denote derivatives with respect to t)

$$\lambda' = 3\lambda^2 + 8\lambda\nu - 48\nu^2$$
$$\nu' = 10\nu^2 \quad .$$

We can immediately integrate the second equation to find

$$\frac{1}{\nu_0} - \frac{1}{\nu} = 10t$$

or

$$\nu(t) = \frac{\nu_0}{1 - 10\nu_0 t} \quad .$$

Now we work on the the beta function involving λ. We try the ansatz $\lambda(t) = A(t)\nu(t)$. Then the first equation above becomes

$$A'\nu + A\nu' = (3A^2 + 8A - 48)\nu^2$$

and plugging in for ν' we get

$$A' = (3A^2 - 2A - 48)\nu \quad .$$

We separate this equation and write it as

$$\int \frac{dA}{3(A - A_+)(A - A_-)} = \int \nu dt$$

with $A_\pm = (1 \pm \sqrt{145})/3$. There are two cases we must consider. If $(A - A_+)(A - A_-) > 0$ this integral is given by

$$\int \frac{dA}{3(A - A_+)(A - A_-)} = \frac{1}{3(A_+ - A_-)} \log\left(\frac{A - A_+}{A - A_-}\right)$$

and if $(AaA_+)(A - A_-) < 0$ it is given by

$$\int \frac{dA}{3(A - A_+)(A - A_-)} = -\frac{1}{\sqrt{145}} \tanh^{-1}\left(\frac{3A - 1}{\sqrt{145}}\right) \quad .$$

We now integrate ν:

$$\int \nu dt = -\frac{1}{10} \log(1 - 10\nu_0 t) = \frac{1}{10} \log\left(\frac{\nu}{\nu_0}\right)$$

where ν_0 is an arbitrary constant of integration. In the case $(A - A_+)(A - A_-) > 0$ we obtain

$$\frac{1}{3(A_+ - A_-)} \log\left(\frac{A - A_+}{A - A_-}\right) = \frac{1}{10} \log\left(\frac{\nu}{\nu_0}\right)$$

or

$$A = \frac{A_+ - A_-(\nu/\nu_0)^{3(A_+ - A_-)/10}}{1 - (\nu/\nu_0)^{3(A_+ - A_-)/10}} \quad .$$

We note that $A \to \infty(-\infty)$ as $\nu \to \nu_0$ from below (above). Also note that as $\nu/\nu_0 \to \infty$, $A \to A_-$. Now, in the case $(A - A_+)(A - A_-) < 0$ we obtain

$$-\frac{1}{\sqrt{145}} \tanh^{-1}\left(\frac{3A - 1}{\sqrt{145}}\right) = \frac{1}{10} \log\left(\frac{\nu}{\nu_0}\right)$$

or

$$A = \frac{1}{3}\left\{1 - \sqrt{145} \tanh\left[\frac{\sqrt{145}}{10} \log\left(\frac{\nu}{\nu_0}\right)\right]\right\} \quad .$$

Now note that as $\nu/\nu_0 \to \infty$, $A \to (1-\sqrt{145})/3 = A_-$. Using these results we can write an equation for λ as a function of g^2:

$$\lambda(g^2) = \begin{cases} \dfrac{A_+ - A_-(g^2/g_0^2)^{3(A_+-A_-)/10}}{1-(g^2/g_0^2)^{3(A_+-A_-)/10}} g^2 \\ \dfrac{1}{3}\left\{1-\sqrt{145}\tanh\left[\dfrac{\sqrt{145}}{10}\log\left(\dfrac{g^2}{g_0^2}\right)\right]\right\} g^2 \end{cases}.$$

For the purposes of graphing these we can express them numerically using $A_+ \approx 4.35$ and $A_- \approx -3.68$:

$$\lambda(g^2) = \begin{cases} \dfrac{4.35 + 3.68(g^2/g_0^2)^{7.22}}{1-(g^2/g_0^2)^{7.22}} g^2 \\ \dfrac{1}{3}\left\{1-12.04\tanh\left[(1.204)\log\left(\dfrac{g^2}{g_0^2}\right)\right]\right\} g^2 \end{cases}.$$

These functions are plotted for $g_0^2 = 2, 3, 5, 7$ along with the characteristics $\lambda = A_\pm g^2$ below. Fabian Schmidt was also kind enough to send us a version of his plot seen below. It bears out the same conclusions.

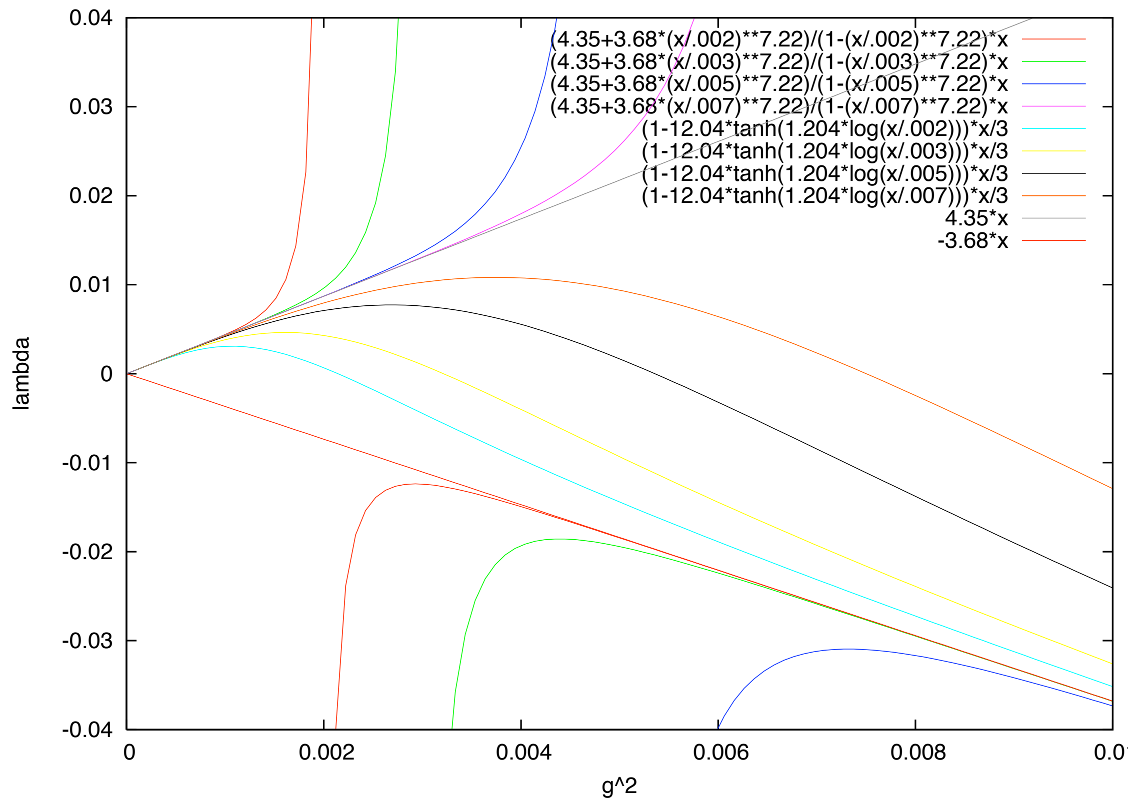


Figure 1: λ vs. g^2 flows. All flows are in the direction $g^2 \to \infty$ (since $\beta_{g^2} > 0$). The values dividing x in the labels are the values of g_0^2. We can see that the first four formulae correspond to $\lambda > A_+ g^2$ or $\lambda < A_- g^2$ while the last four correspond to $A_- g^2 < \lambda < A_+ g^2$.

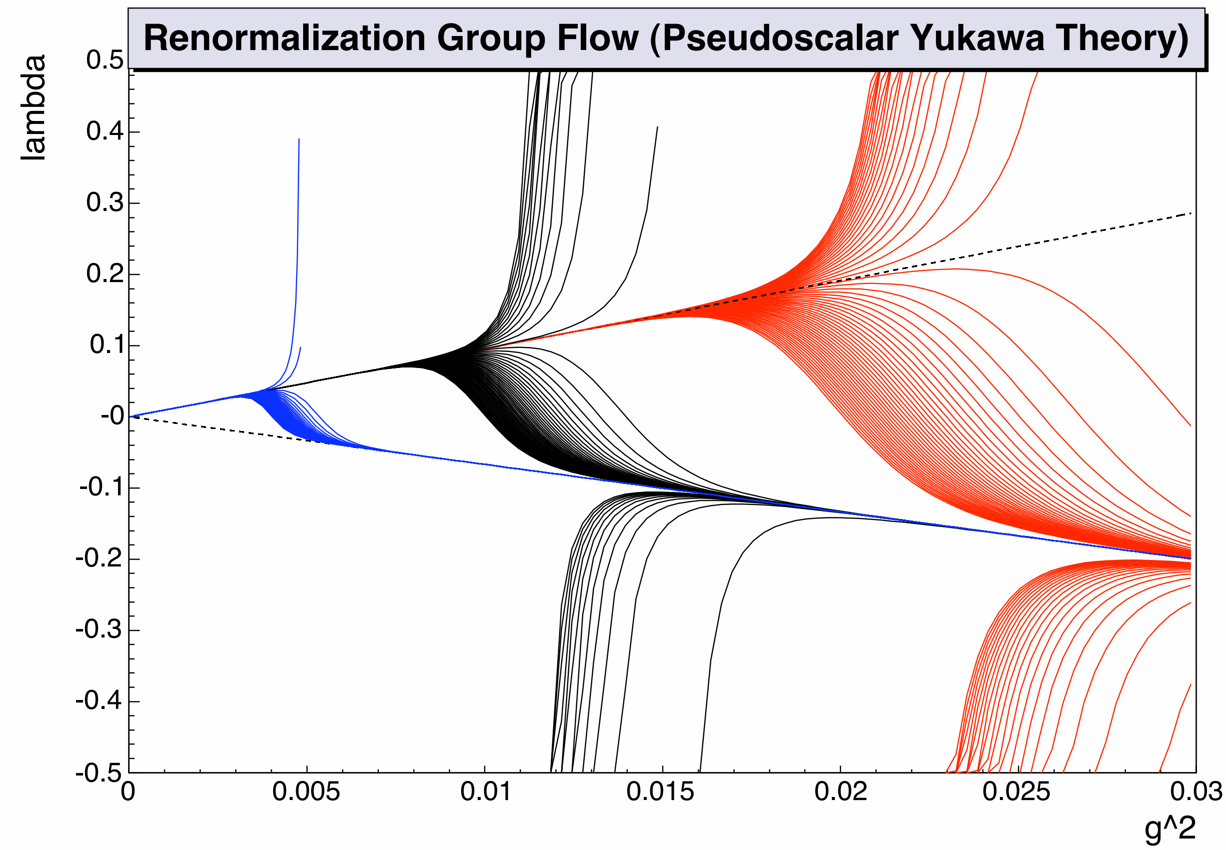


Figure 2: λ vs. g^2 flows by Fabian Schmidt.