QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEMS DUE TUESDAY, February 21

Problem in text	**Subject**
12–2)	Beta function in Gross-Neveu model

As in the problem for the previous week, you will find the paper by D. Gross and A. Neveu, Phys. Rev. D **10**, 3235–3253, helpful. The problem due Tuesday, February 28 will be 12–1 (beta functions in Yukawa theory).

Solutions — Problem Set 6

David McKeen – February 21, 2006

Problem 12–2 As seen in the last problem, the Gross-Neveu model can be described by the Lagrangian

$$\mathcal{L} = \bar{\psi}_i i \partial\!\!\!/ \psi_i - g\sigma\bar{\psi}_i\psi_i - \frac{1}{2}\sigma^2 \quad .$$

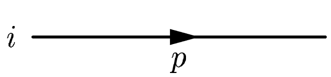
We rescale the fields as $\psi_i \to \sqrt{Z_\psi}\psi_i$ and $\sigma \to \sqrt{Z_\sigma}\sigma$. Then we find

$$\mathcal{L} = Z_\psi\bar{\psi}_i i \partial\!\!\!/ \psi_i - g\sqrt{Z_\sigma}Z_\psi\sigma\bar{\psi}_i\psi_i - \frac{1}{2}Z_\sigma\sigma^2 \quad .$$

We now define the counterterms $\delta_\psi = Z_\psi - 1$, $\delta_\sigma = Z_\sigma - 1$, and $\delta_g = g\sqrt{Z_\sigma}Z_\psi - g$. The the Lagrangian can be written

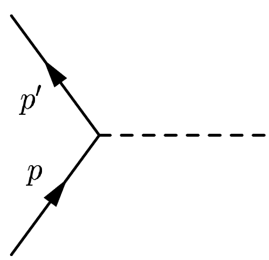
$$\mathcal{L} = \bar{\psi}_i i \partial\!\!\!/ \psi_i - g\sigma\bar{\psi}_i\psi_i - \frac{1}{2}\sigma^2 + \delta_\psi\bar{\psi}_i i \partial\!\!\!/ \psi_i - \delta_g\sigma\bar{\psi}_i\psi_i - \frac{1}{2}\delta_\sigma\sigma^2 \quad .$$

This gives us the Feynman rules



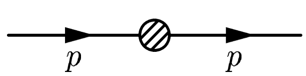
$= \dfrac{i\delta_{ij}\, p\!\!\!/}{p^2}$

$= -i$

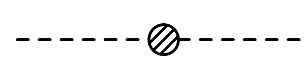


$= -ig \quad .$

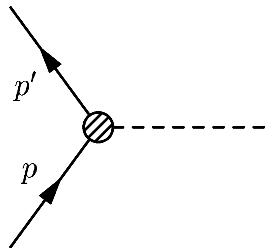
The counterterm rules are



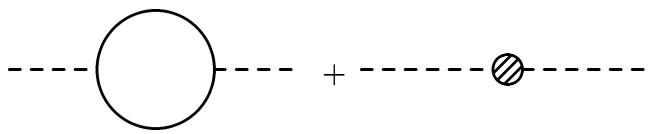
$= i\delta_{ij}\delta_\psi$



$= -i\delta_\sigma$



The lowest order correction to the scalar propagator is given by



The amplitude for the loop diagram is

$$
\begin{aligned}
-iM^2 &= (-ig)^2 N \int \frac{d^2k}{(2\pi)^2} \mathrm{tr} \frac{\not k(\not k + \not p)}{k^2(k+p)^2} \\
&= -g^2 N \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{tr}[\not k(\not k + \not p)]}{(k^2 + 2xk\cdot p + xp^2)^2} \\
&= -g^2 N \int_0^1 dx \int \frac{d^2\ell}{(2\pi)^2} \frac{\mathrm{tr}[(\not \ell - x\not p)(\not \ell - (x-1)\not p)]}{(\ell^2 - \Delta)^2}
\end{aligned}
$$

with $\ell = k + xp$ and $\Delta = x(x-1)p^2$. Then, dropping terms linear in ℓ and noting that $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}$ in 2 dimensions,

$$
-iM^2 = -2g^2 N \int_0^1 dx \int \frac{d^2\ell}{(2\pi)^2} \frac{\ell^2 + \Delta}{(\ell^2 - \Delta)^2} \quad .
$$

We use equation (A.45) of Peskin and Schroeder to carry out the integral and find that

$$
\begin{aligned}
-iM^2 &= -2g^2 N \int_0^1 dx \left[-\frac{i}{4\pi}\left(\frac{2}{\epsilon}\right)\right] \\
&= \frac{ig^2 N}{2\pi}\left(\frac{2}{\epsilon}\right) \quad .
\end{aligned}
$$

To cancel this divergence we require that

$$
\frac{ig^2 N}{2\pi}\left(\frac{2}{\epsilon}\right) - i\delta_\sigma = 0
$$

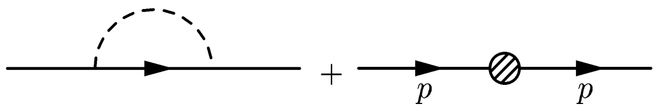
or

$$\delta_\sigma \sim \frac{g^2 N}{2\pi}\left(\frac{2}{\epsilon}\right) \quad .$$

Since $\delta_\sigma \sim B_\sigma \log(\lambda^2/M^2) = B_\sigma(2/\epsilon)$ we identify

$$A_\sigma = \frac{g^2 N}{2\pi} \quad .$$

The correction to the fermion propagator is



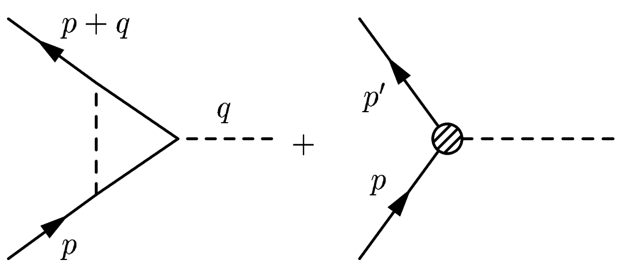
The first diagram gives

$$-i\Sigma = (-ig)^2(-i)\int \frac{d^2k}{(2\pi)^2}\frac{i\,\not{k}}{k^2} = 0$$

since the integrand is odd. Thus we don't need the counterterm to cancel any divergences and

$$A_\psi = 0 \quad .$$

The correction to the vertex is



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The diagram on the left gives

$$\begin{aligned}\bar{u}(p+q)\delta\Gamma u(p) &= (-ig)^3(-i)\bar{u}(p+q)\int \frac{d^2k}{(2\pi)^2}\frac{i(\not{k}+\not{q})i\,\not{k}}{(k+q)^2k^2}u(p)\\ &= -g^3\bar{u}(p+q)\int \frac{d^2k}{(2\pi)^2}\frac{(\not{k}+\not{q})\,\not{k}}{(k+q)^2k^2}u(p)\\ &= -g^3\bar{u}(p+q)\int \frac{d^2k}{(2\pi)^2}\frac{(\not{k}+\not{q})\,\not{k}}{(k+q)^2k^2}u(p) \quad .\end{aligned}$$

The divergent part of this integral is

$$\begin{aligned}\delta\Gamma &\sim -g^3 \int \frac{d^2k}{(2\pi)^2} \frac{\not k \not k}{(k+q)^2 k^2} \\ &= -g^3 \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k+q)^2} \\ &= -g^3 \int \frac{d^2\ell}{(2\pi)^2} \frac{1}{\ell^2} \\ &= -g^3 \left[-\frac{i}{4\pi} \left(\frac{2}{\epsilon} \right) \right] \\ &= \frac{ig^3}{4\pi} \left(\frac{2}{\epsilon} \right) \quad .\end{aligned}$$

To cancel this divergence we find that

$$-i\delta_g \sim -\frac{ig^3}{4\pi} \left(\frac{2}{\epsilon} \right)$$

or

$$\delta_g \sim \frac{g^3}{4\pi} \left(\frac{2}{\epsilon} \right) = -B_g \left(\frac{2}{\epsilon} \right)$$

and

$$B_g = -\frac{g^3}{4\pi} \quad .$$

The beta function is given by

$$\beta(g) = -2B_g - g(A_\sigma + 2A_\psi)$$

where the factor of 2 multiplying A_ψ comes from the fact that the vertex involving g involves 2 ψ fields. Plugging our values in we find

$$\beta(g) = -\frac{g^3}{2\pi}(N-1) \quad .$$

We see that the theory is asymptotically free for 2 or more fermion fields.