QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEMS DUE TUESDAY, February 14

Problem in text	**Subject**
11–3(a)-(e)	Gross-Neveu model

Note that you are being given an extra week for this problem, since we will be reaching some of the relevant material only in the week before it is due. You will find the paper by D. Gross and A. Neveu, Phys. Rev. D **10**, 3235–3253, very illuminating. It contains some material on the renormalization group which we will cover only later, but that is not relevant for the results of Problem 11-3.

Solutions — Problem Set 5

David McKeen – February 14, 2006

Problem 11–3

(a) The Lagrangian is

$$\mathcal{L} = \bar{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g^2 \left(\bar{\psi}_i \psi_i\right)^2$$

If we take $\psi_i \to \gamma_5 \psi_i$ then $\bar{\psi}_i$ transforms as

$$\begin{aligned}\bar{\psi}_i &\to \left(\gamma_5 \psi_i\right)^\dagger \gamma^0 = \psi_i^\dagger \gamma_5^\dagger \gamma^0 = \psi_i^\dagger \gamma_5 \gamma^0 \\ &= -\bar{\psi}_i \gamma_5 \quad .\end{aligned}$$

Under this tranformation the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &\to -\bar{\psi}_i \gamma_5 i \not{\partial} \gamma_5 \psi_i + \frac{1}{2} g^2 \left(-\bar{\psi}_i \gamma_5^2 \psi_i\right)^2 \\ &= \bar{\psi}_i i \not{\partial} \gamma_5^2 \psi_i + \frac{1}{2} g^2 \left(\bar{\psi}_i \psi_i\right)^2 \\ &= \bar{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g^2 \left(\bar{\psi}_i \psi_i\right)^2 \\ &= \mathcal{L} \quad .\end{aligned}$$

Thus the theory is invariant under this chiral transformation. Note that a mass term is not invariant:

$$\bar{\psi}_i \psi_i \to -\bar{\psi}_i \gamma_5^2 \psi_i = -\bar{\psi}_i \psi_i \quad .$$

(b) In 2 dimensions the Lagrangian must have (mass) dimension 2 so that the action is dimensionless. To obtain the dimensionality of the ψ_i field we examine the kinetic term. The derivative has dimension 1. Then ψ_i must have dimension 1/2. Now we take a look at the interaction term. Having determined the dimensionality of ψ_i we see that $(\bar{\psi}_i \psi)^2$ has dimension 2. Therefore the coupling g^2 is dimensionless and the theory is renormalizable.

(c) The functional integral can be written

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left\{ i \int d^4x \; \left[\bar{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g^2 \left(\bar{\psi}_i \psi_i\right)^2 \right] \right\} \quad .$$

We now introduce an irrelevant constant in the form of a Gaussian integral over a scalar field to write the functional integral as

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi \exp\left\{i\int d^4x \ \left[\bar{\psi}_i i \not{\partial}\psi_i + \frac{1}{2}g^2\left(\bar{\psi}_i\psi_i\right)^2 - \frac{1}{2g^2}\phi^2\right]\right\} \quad .$$

Now we make the change of variable $\phi \to \sigma + g^2\bar{\psi}_i\psi_i$ which is just a shift so the Jacobian is 1. The functional integral becomes

$$\begin{aligned}
&\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\left\{i\int d^4x \ \left[\bar{\psi}_i i \not{\partial}\psi_i + \frac{1}{2}g^2\left(\bar{\psi}_i\psi_i\right)^2 - \frac{1}{2g^2}\left(\sigma + g^2\bar{\psi}_i\psi_i\right)^2\right]\right\} \\
&= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\left\{i\int d^4x \ \left[\bar{\psi}_i i \not{\partial}\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right]\right\}
\end{aligned}$$

(d) It will be helpful to take $\sigma \to g\sigma$. The Lagrangian becomes

$$\mathcal{L} = \bar{\psi}_i i \not{\partial}\psi_i - g\sigma\bar{\psi}_i\psi_i - \frac{1}{2}\sigma^2 \quad . \tag{1}$$

We use this form of the Lagrangian to write down the generating functional of the theory:

$$Z[J] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\left\{i\int d^4x \ \left[\bar{\psi}_i i \not{\partial}\psi_i - g\sigma\bar{\psi}_i\psi_i - \frac{1}{2}\sigma^2 + J\sigma\right]\right\} \quad .$$

We now expand around the classical scalar field: $\sigma = \sigma_{\rm cl} + \phi$. Then the generating functional becomes

$$\begin{aligned}
Z[J] = \exp&\left[i\int d^4x \ \left(-\frac{1}{2}\sigma_{\rm cl}^2 + J\sigma_{\rm cl}\right)\right]\int \mathcal{D}\phi \exp\left[i\int d^4x \ \left(-\frac{1}{2}\phi^2 - \sigma_{\rm cl}\phi + J\phi\right)\right] \\
&\times \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int d^4x \ \left(\bar{\psi}_i i \not{\partial}\psi_i - g\sigma_{\rm cl}\bar{\psi}_i\psi_i - g\phi\bar{\psi}_i\psi_i\right)\right] \quad .
\end{aligned}$$

As explained in section 11.4 of Peskin and Schroeder, counterterms in J will cancel tadpole diagrams so that we can ignore terms linear in ϕ. The integral over ϕ then becomes an unimportant constant. We also perform the integral over the fermionic fields and the functional integral (up to meaningless constants) becomes

$$\begin{aligned}
Z[J]\big|_{J=0} &= \exp\left[i\int d^4x \ \left(-\frac{1}{2}\sigma_{\rm cl}^2\right)\right]\left[\det\left(i\not{\partial} - g\sigma_{\rm cl}\right)\right]^N \\
&= \exp\left[i\int d^4x \ \left(-\frac{1}{2}\sigma_{\rm cl}^2\right)\right]\exp\left[N \ {\rm Tr}\log\left(i\not{\partial} - g\sigma_{\rm cl}\right)\right] \quad .
\end{aligned}$$

If we now look back to the Lagrangian in Eq. (1) and take $\sigma \to Z_\sigma \sigma$ and then define $\delta_\sigma = Z_\sigma - 1$ we see that we will find a term $\delta_\sigma \sigma^2/2$ in the counterterm Lagrangian. The effective action is then given by

$$\begin{aligned}\Gamma[\sigma_{\rm cl}] &= -i \log Z[J=0] \\ &= \int d^4x \left(-\frac{1}{2}\sigma_{\rm cl}^2 - \frac{1}{2}\delta_\sigma \sigma_{\rm cl}^2\right) - iN \ {\rm Tr} \log \left(i \not{\partial} - g\sigma_{\rm cl}\right) \quad .\end{aligned}$$

We now work on taking this trace:

$$\mathrm{Tr} \log \left(i \not{\partial} - \sigma_{\rm cl}\right) = (VT) \int \frac{d^2p}{(2\pi)^2} \mathrm{tr} \left[\log \left(\not{p} - g\sigma_{\rm cl}\right)\right] \quad .$$

The trace of the logarithm of a matrix is the sum of the logarithms of its eigenvalues. Using

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then

$$\not{p} - g\sigma_{\rm cl} = \begin{pmatrix} -g\sigma_{\rm cl} & -i(p_0 - p_1) \\ i(p_0 + p_1) & -g\sigma_{\rm cl} \end{pmatrix} \quad .$$

This has eigenvalues $-g\sigma_{\rm cl} + \sqrt{p^2}$ and $-g\sigma_{\rm cl} - \sqrt{p^2}$. Then

$$\begin{aligned}\mathrm{tr} \left[\log \left(\not{p} - g\sigma_{\rm cl}\right)\right] &= \log(-g\sigma_{\rm cl} + \sqrt{p^2}) + \log(-g\sigma_{\rm cl} - \sqrt{p^2}) \\ &= \log(g^2\sigma_{\rm cl}^2 - p^2) \quad .\end{aligned}$$

This gives

$$\mathrm{Tr} \log \left(i \not{\partial} - g\sigma_{\rm cl}\right) = (VT) \int \frac{d^2p}{(2\pi)^2} \log(g^2\sigma_{\rm cl}^2 - p^2) \quad .$$

We will do this integral using dimensional regularization.

$$\begin{aligned}\int \frac{d^dp}{(2\pi)^d} \log(g^2\sigma_{\rm cl}^2 - p^2) &= i \int \frac{d^dp_E}{(2\pi)^d} \log(p_E^2 + g^2\sigma_{\rm cl}^2) \\ &= -i\frac{\partial}{\partial\alpha} \int \frac{d^dp_E}{(2\pi)^d} \frac{1}{(p_E^2 + g^2\sigma_{\rm cl}^2)^\alpha}\bigg|_{\alpha=0} \\ &= -i\frac{\partial}{\partial\alpha} \left[\frac{(g^2\sigma_{\rm cl}^2)^{d/2-\alpha}}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)}\right]_{\alpha=0} \\ &= -i\left(g^2\frac{\sigma_{\rm cl}^2}{4\pi}\right)^{d/2} \Gamma\left(-\frac{d}{2}\right) \quad .\end{aligned}$$

The gamma function has a pole at -1. To examine this behavior we will set $\epsilon = d - 2$ and take $\epsilon \to 0$. Then

$$\begin{aligned}
\Gamma\left(-\frac{d}{2}\right) &= \Gamma\left(\frac{\epsilon}{2} - 1\right) \\
&= \frac{1}{\epsilon/2 - 1}\Gamma\left(\frac{\epsilon}{2}\right) \\
&\approx -\left(1 + \frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}\right) \\
&\approx -\left(1 + \frac{\epsilon}{2}\right)\left(\frac{2}{\epsilon} - \gamma\right) \\
&\approx -\left(1 + \frac{2}{\epsilon} - \gamma\right) \quad .
\end{aligned}$$

Also as $\epsilon \to 0$

$$\begin{aligned}
\left(g^2 \frac{\sigma_{\text{cl}}^2}{4\pi}\right)^{d/2} &= \left(\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right)\left(\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right)^{-\epsilon/2} \\
&\to \left(\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right)\left(1 - \frac{\epsilon}{2}\log\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right) \quad .
\end{aligned}$$

Then we see that

$$\int \frac{d^d p}{(2\pi)^d}\log(g^2\sigma_{\text{cl}}^2 - p^2) \to i\left(\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right)\left(1 - \log\frac{g^2\sigma_{\text{cl}}^2}{4\pi} + \frac{2}{\epsilon} - \gamma\right) \quad .$$

We collect these results and write

$$\begin{aligned}
\Gamma[\sigma_{\text{cl}}] &= -(VT)\left[\frac{1}{2}\sigma_{\text{cl}}^2 + \frac{1}{2}\delta_\sigma\sigma_{\text{cl}}^2 - N\left(\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right)\left(1 - \log\frac{g^2\sigma_{\text{cl}}^2}{4\pi} + \frac{2}{\epsilon} - \gamma\right)\right] \\
&= -(VT)V_{\text{eff}}(\sigma_{\text{cl}}) \quad .
\end{aligned}$$

If we now set

$$\delta_\sigma = \frac{g^2 N\sigma_{\text{cl}}^2}{2\pi}\left(\log 4\pi - \log M^2 + \frac{2}{\epsilon} - \gamma\right)$$

(i.e. use modified minimal subtraction) where M^2 is the scale at which we define the theory the effective action becomes

$$V_{\text{eff}}(\sigma_{\text{cl}}) = \frac{1}{2}\sigma_{\text{cl}}^2 - N\left(\frac{g^2\sigma_{\text{cl}}^2}{4\pi}\right)\left(1 - \log\frac{g^2\sigma_{\text{cl}}^2}{M^2}\right) \quad .$$

(e) Minimizing this potential:

$$0 = \frac{\partial V_{\text{eff}}}{\partial \sigma_{\text{cl}}} = \sigma_{\text{cl}} \left(1 + \frac{g^2 N}{2\pi} \log \frac{g^2 \sigma_{\text{cl}}^2}{M^2} \right) \quad .$$

This gives

$$g\sigma_{\text{cl}} = \pm M e^{-\pi/g^2 N} \quad .$$

Then, for any nonzero value of Λ the fermion fields acquire a mass which breaks the chiral symmetry of the original Lagrangian.