QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEMS DUE TUESDAY, JANUARY 31

Problem in text	**Subject**
10–1 (a)	Furry's Theorem
10–1 (b)	Finiteness of light-light scattering

Hints: (a) The identity $\Gamma_C \gamma^\mu \Gamma_C^{-1} = -\gamma^{\mu T}$, where $\Gamma_C = \gamma^0 \gamma^2$, may be useful here.

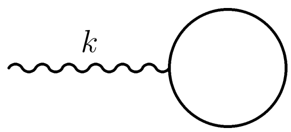
(b) It is enough to study the leading behavior of the trace (in powers of loop momentum) when summing over six distinct diagrams.

Solutions — Problem Set 4

David McKeen – January 31, 2006

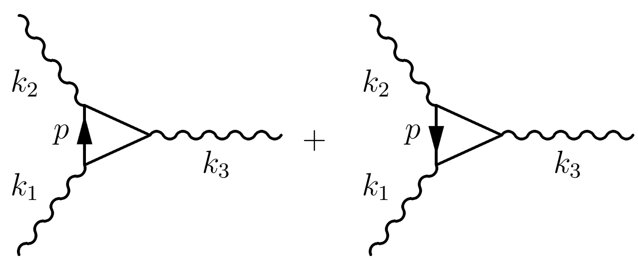
Problem 10–1

(a) Note that in these solutions all photon momenta will be directed towards the vertex. The one point photon vertex is

$$i\mathcal{M}^{\mu} =$$ 

$$\begin{aligned}
&= -\int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left[\frac{i(\not{p}+m)}{p^2-m^2}(-ie\gamma^{\mu})\right] \\
&= 4ie\int \frac{d^4p}{(2\pi)^4}\frac{p^{\mu}}{p^2-m^2} \\
&= 0 \quad .
\end{aligned}$$

The three photon vertex is given by

$$i\mathcal{M}^{\mu_1\mu_2\mu_3} =$$ 

$$= i\mathcal{M}_1^{\mu_1\mu_2\mu_3} + i\mathcal{M}_2^{\mu_1\mu_2\mu_3} \quad .$$

We can write the amplitudes as

$$i\mathcal{M}_1^{\mu_1\mu_2\mu_3} = -e^3\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+m)\gamma^{\mu_3}(\not{p}+\not{k}_2+\not{k}_3+m)\right]}{[p^2-m^2]\left[(p+k_2)^2-m^2\right]\left[(p+k_2+k_3)^2-m^2\right]}$$

and

$$i\mathcal{M}_2^{\mu_1\mu_2\mu_3} = -e^3 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}-\not{k}_2+m)\gamma^{\mu_3}(\not{p}-\not{k}_2-\not{k}_3+m)\right]}{[p^2-m^2]\,[(p-k_2)^2-m^2]\,[(p-k_2-k_3)^2-m^2]} \quad .$$

where we have reversed the order of all the gamma matrices in the last trace. We will work on this trace. Consider $\Gamma_C = \gamma^0\gamma^2$. It's easily seen that $\Gamma_C^2 = 1$ and $\Gamma_C\gamma^\mu\Gamma_C = -(\gamma^\mu)^T$. We insert factors of Γ_C^2 after each Dirac matrix in the trace and use the cyclicity of the trace to move the last Γ_C to the front. We see

$$\begin{aligned}
&\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}-\not{k}_2+m)\gamma^{\mu_3}(\not{p}-\not{k}_2-\not{k}_3+m)\right] \\
&\quad = \mathrm{tr}\left[\Gamma_C\gamma^{\mu_1}\Gamma_C^2(\not{p}+m)\Gamma_C^2\gamma^{\mu_2}\Gamma_C^2(\not{p}-\not{k}_2+m)\Gamma_C^2\gamma^{\mu_3}\Gamma_C^2(\not{p}-\not{k}_2-\not{k}_3+m)\Gamma_C\right] \\
&\quad = \mathrm{tr}\left[(\gamma^{\mu_1})^T(\not{p}-m)^T(\gamma^{\mu_2})^T(\not{p}-\not{k}_2-m)^T(\gamma^{\mu_3})^T(\not{p}-\not{k}_2-\not{k}_3-m)^T\right] \\
&\quad = \mathrm{tr}\left[(\not{p}-\not{k}_2-\not{k}_3-m)\gamma^{\mu_3}(\not{p}-\not{k}_2-m)\gamma^{\mu_2}(\not{p}-m)\gamma^{\mu_1}\right] \\
&\quad = \mathrm{tr}\left[\gamma^{\mu_1}(\not{p}-m)\gamma^{\mu_2}(\not{p}-\not{k}_2-m)\gamma^{\mu_3}(\not{p}-\not{k}_2-\not{k}_3-m)\right] \quad .
\end{aligned}$$

We can then write

$$i\mathcal{M}_2^{\mu_1\mu_2\mu_3} = -e^3 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}-m)\gamma^{\mu_2}(\not{p}-\not{k}_2-m)\gamma^{\mu_3}(\not{p}-\not{k}_2-\not{k}_3-m)\right]}{[p^2-m^2]\,[(p-k_2)^2-m^2]\,[(p-k_2-k_3)^2-m^2]} \quad .$$

We make the change of variable in the integral $p \to -p$. It becomes

$$\begin{aligned}
i\mathcal{M}_2^{\mu_1\mu_2\mu_3} &= -e^3(-1)^3 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+m)\gamma^{\mu_3}(\not{p}+\not{k}_2+\not{k}_3+m)\right]}{[p^2-m^2]\,[(p+k_2)^2-m^2]\,[(p+k_2+k_3)^2-m^2]} \quad . \\
&= -i\mathcal{M}_1^{\mu_1\mu_2\mu_3} \quad .
\end{aligned}$$

From this we see that the three-point photon vertex function vanishes.

We now deal with the n-point photon vertex. There are $(n-1)!$ diagrams that contribute. It is easy to convince oneself that for each of these diagrams there is one other that differs only in the direction of the internal electron momentum flow. That is, there are $(n-1)!/2$ topologically distinct diagrams. We will show that there is a simple relation between the diagrams that differ only under the direction of electron momentum. Consider a generic n-point diagram. It can be written (with an appropriate labelling of external momenta) as

$$i\mathcal{M}_1^{\{\mu_n\}} = -e^n \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+m)\ldots\gamma^{\mu_n}(\not{p}+\not{k}_2+\cdots+\not{k}_n+m)\right]}{[p^2-m^2]\left[(p+k_2)^2-m^2\right]\ldots\left[(p+k_2+\cdots+k_n)^2-m^2\right]} \quad .$$

The diagram that differs under the direction of the electron momentum can then be seen to be (with the order of gamma matrices reversed):

$$i\mathcal{M}_2^{\{\mu_n\}} = -e^n \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}-\not{k}_2+m)\ldots\gamma^{\mu_n}(\not{p}-\not{k}_2-\cdots-\not{k}_n+m)\right]}{[p^2-m^2]\left[(p-k_2)^2-m^2\right]\ldots\left[(p-k_2-\cdots-k_n)^2-m^2\right]} \quad .$$

We can then manipulate the trace in the second amplitude as we did above to find that

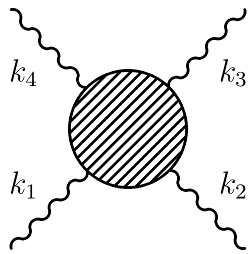
$$i\mathcal{M}_2^{\{\mu_n\}} = -e^n \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}-m)\gamma^{\mu_2}(\not{p}-\not{k}_2-m)\ldots\gamma^{\mu_n}(\not{p}-\not{k}_2-\cdots-\not{k}_n-m)\right]}{[p^2-m^2]\left[(p-k_2)^2-m^2\right]\ldots\left[(p-k_2-\cdots-k_n)^2-m^2\right]} \quad .$$

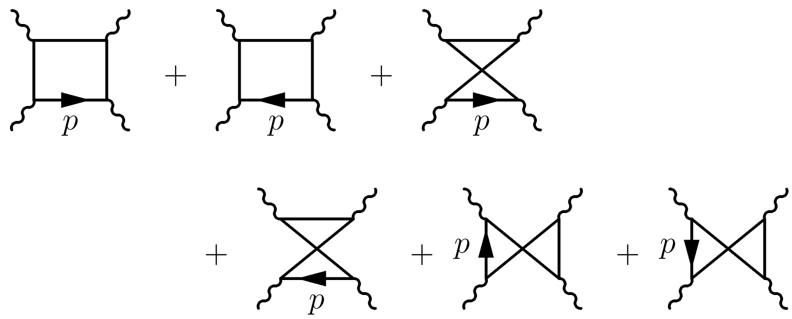
If we again take $p \to -p$ we see that

$$\begin{aligned} i\mathcal{M}_2^{\{\mu_n\}} &= -e^n(-1)^n \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+m)\ldots\gamma^{\mu_n}(\not{p}+\not{k}_2+\cdots+\not{k}_n+m)\right]}{[p^2-m^2]\left[(p+k_2)^2-m^2\right]\ldots\left[(p+k_2+\cdots+k_n)^2-m^2\right]} \\ &= (-1)^n i\mathcal{M}_1^{\{\mu_n\}} \quad . \end{aligned}$$

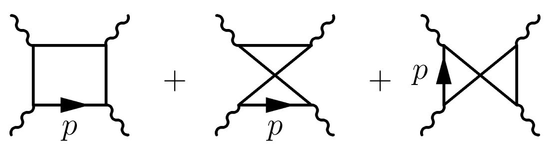
Thus we see that if n is odd each diagram is canceled by the corresponding one with opposite loop momentum and the total amplitude is zero.

(b) The four photon interaction amplitude is given to lowest order by six diagrams:

$i\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4} =$ 

$=$ 

The argument in part (a) allows us to only consider 3 diagrams:

$\frac{1}{2}\left(i\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4}\right) =$ 

$$= i\mathcal{M}_1^{\{\mu_4\}} + i\mathcal{M}_2^{\{\mu_4\}} + i\mathcal{M}_3^{\{\mu_4\}} \quad .$$

We write down the amplitudes

$$i\mathcal{M}_1^{\{\mu_4\}} = -e^4 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+m)\gamma^{\mu_3}(\not{p}+\not{k}_2+\not{k}_3+m)\gamma^{\mu_4}(\not{p}+\not{k}_2+\not{k}_3+\not{k}_4+m)\right]}{\left[p^2-m^2\right]\left[(p+k_2)^2-m^2\right]\left[(p+k_2+k_3)^2-m^2\right]\left[(p+k_2+k_3+k_4)^2-m^2\right]}$$
$$i\mathcal{M}_2^{\{\mu_4\}} = -e^4 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+m)\gamma^{\mu_4}(\not{p}+\not{k}_2+\not{k}_4+m)\gamma^{\mu_3}(\not{p}+\not{k}_2+\not{k}_3+\not{k}_4+m)\right]}{\left[p^2-m^2\right]\left[(p+k_2)^2-m^2\right]\left[(p+k_2+k_4)^2-m^2\right]\left[(p+k_2+k_3+k_4)^2-m^2\right]}$$
$$i\mathcal{M}_1^{\{\mu_4\}} = -e^4 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^{\mu_1}(\not{p}+m)\gamma^{\mu_4}(\not{p}+\not{k}_4+m)\gamma^{\mu_2}(\not{p}+\not{k}_2+\not{k}_4+m)\gamma^{\mu_3}(\not{p}+\not{k}_2+\not{k}_3+\not{k}_4+m)\right]}{\left[p^2-m^2\right]\left[(p+k_4)^2-m^2\right]\left[(p+k_2+k_4)^2-m^2\right]\left[(p+k_2+k_3+k_4)^2-m^2\right]} \quad .$$

The potentially UV divergent term (which we denote $i\mathcal{M}_{\mathrm{div}}^{\{\mu_4\}}$) can then be written as

$$i\mathcal{M}_{\rm div}^{\{\mu_4\}} = -e^4 \int \frac{d^4p}{(2\pi)^4} p^{-8} \ {\rm tr}\left[\gamma^{\mu_1} \not{p} \gamma^{\mu_2} \not{p} \gamma^{\mu_3} \not{p} \gamma^{\mu_4} \not{p} + \gamma^{\mu_1} \not{p} \gamma^{\mu_2} \not{p} \gamma^{\mu_4} \not{p} \gamma^{\mu_3} \not{p} + \gamma^{\mu_1} \not{p} \gamma^{\mu_4} \not{p} \gamma^{\mu_2} \not{p} \gamma^{\mu_3} \not{p}\right] \quad .$$

We can pull out a factor of $p_\alpha p_\beta p_\gamma p_\delta$ which will be replaced by $p^4(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma})/24$ since the integral is symmetric (Eq. (A.42) of Peskin and Schroeder). A generic term in the trace becomes

$$\begin{aligned}
(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma})(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho\gamma^\gamma\gamma^\sigma\gamma^\delta) &= \gamma^\mu\gamma^\alpha\gamma^\nu\gamma_\alpha\gamma^\rho\gamma^\gamma\gamma^\sigma\gamma_\gamma \\
&+ \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho\gamma_\alpha\gamma^\sigma\gamma_\beta + \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho\gamma^\beta\gamma^\sigma\gamma_\alpha \\
&= 4\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - 2\gamma^\mu\gamma^\rho\gamma^\beta\gamma^\nu\gamma^\sigma\gamma_\beta - 2\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\alpha \\
&= 4\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - 8g^{\nu\sigma}\gamma^\mu\gamma^\rho + 4\gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu \quad .
\end{aligned}$$

We now note that

$$\gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu = -\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + 2g^{\rho\sigma}\gamma^\mu\gamma^\nu - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho + 2g^{\nu\rho}\gamma^\mu\gamma^\sigma$$

so that the term becomes

$$8\left(g^{\rho\sigma}\gamma^\mu\gamma^\nu + g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho\right) \tag{1}$$

To get each term in the trace we take $\{\mu,\nu,\rho,\sigma\} \to \{\mu_1,\mu_2,\mu_3,\mu_4\}, \{\mu_1,\mu_2,\mu_4,\mu_3\}, \{\mu_1,\mu_4,\mu_2,\mu_3\}$ respectively. The amplitude becomes

$$\begin{aligned}
i\mathcal{M}_{\rm div}^{\{\mu_4\}} = -\frac{e^4}{3}\int \frac{d^4p}{(2\pi)^4} p^{-4} \ {\rm tr}\,(& g^{\mu_3\mu_4}\gamma^{\mu_1}\gamma^{\mu_2} + g^{\mu_2\mu_3}\gamma^{\mu_1}\gamma^{\mu_4} - 2g^{\mu_2\mu_4}\gamma^{\mu_1}\gamma^{\mu_3} \\
& + g^{\mu_4\mu_3}\gamma^{\mu_1}\gamma^{\mu_2} + g^{\mu_2\mu_4}\gamma^{\mu_1}\gamma^{\mu_3} - 2g^{\mu_2\mu_3}\gamma^{\mu_1}\gamma^{\mu_4} \\
& + g^{\mu_2\mu_3}\gamma^{\mu_1}\gamma^{\mu_4} + g^{\mu_4\mu_2}\gamma^{\mu_1}\gamma^{\mu_3} - 2g^{\mu_4\mu_3}\gamma^{\mu_1}\gamma^{\mu_2}) \quad .
\end{aligned}$$

The cancellation between the terms is now manifest and we see that

$$i\mathcal{M}_{\text{div}}^{\{\mu_4\}} = 0$$

as we hoped.