QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEMS DUE TUESDAY, JANUARY 24

Problem in text	**Subject**
9–1 (a,b)	Charged scalar field

You should be aware of the result for the last part of Problem 9–1 (c): It was derived derived in class on Thursday, January 12 using a three-line calculation of the imaginary part of $\Pi^{\mu\nu}(q)$ for a scalar particle.

Additional problem: For a class of gauges, the photon propagator may be written as in Eq. (9.58) of the text:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \quad . \tag{1}$$

The Feynman gauge corresponds to $\xi = 1$ and the Landau gauge corresponds to $\xi = 0$. Show that the second-order contribution to the vertex function, $\Gamma_2^\mu(p', p)$, is finite in the Landau gauge.

Solutions — Problem Set 3

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Problem 9–1

(a) We start with the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^* (D^\mu\phi) - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}^2 + \phi^*(-\partial^2 - m^2)\phi - ieA_\mu (\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + e^2A^2\phi^*\phi \\
&= \mathcal{L}_A + \mathcal{L}_\phi + \mathcal{L}_{\text{int}} \quad .
\end{aligned}
$$

As in section 9.2 of Peskin and Schroeder we define

$$
Z[J, J^*] \equiv \int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left[i \int d^4x[\mathcal{L} + J^*(x)\phi(x) + J(x)\phi^*(x)]\right] \quad .
$$

It's clear then that

$$
\langle 0|\mathrm{T}\phi^*(x_1)\phi(x_2)|0\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J^*(x_2)}\right) Z[J, J^*]\Big|_{J,J^*=0}
$$

To find the scalar propagator we need only consider the part of the Lagrangian not involving A_μ. We follow Peskin and Schroeder and define

$$
\begin{aligned}
\phi'(x) &= \phi(x) - i\int d^4y D_F(x-y)J(y) \\
&= \phi + (-\partial^2 - m^2)^{-1}J
\end{aligned}
$$

and

$$
\begin{aligned}
\phi'^*(x) &= \phi^*(x) - i\int d^4y D_F(x-y)J^*(y) \\
&= \phi^* + (-\partial^2 - m^2)^{-1}J^* \quad .
\end{aligned}
$$

Then

$$\int d^4x \; [\mathcal{L}_\phi + J\phi + J^*\phi^*] = \int d^4x \; \left[\phi'^*(-\partial^2 - m^2)\phi' - J^*\phi' - J\phi'^* + J^*(-\partial^2 - m^2)^{-1}J \right.$$
$$\left. + J\phi'^* - J(-\partial^2 - m^2)^{-1}J^* + J^*\phi' - J^*(-\partial^2 - m^2)^{-1}J\right]$$
$$= \int d^4x \; \left[\phi'^*(-\partial^2 - m^2)\phi' - J^*(-\partial^2 - m^2)^{-1}J\right]$$
$$= \mathcal{L}_{\phi'} + i\int d^4x \; d^4y \; J^*(x)D_F(x-y)J(y) \quad .$$

We shift integration variables in the functional integral from ϕ to ϕ' and ϕ^* to ϕ'^*.

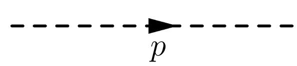
Since this is just a shift the jacobian is 1. We get that

$$Z[J, J^*] = Z_0 \exp\left[-\int d^4x \; d^4y \; J^*(x)D_F(x-y)J(y)\right] \quad .$$

We see that

$$\frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J^*(x_2)}\right) Z[J, J^*]\bigg|_{J,J^*=0}$$
$$= -\frac{1}{Z_0}\frac{\delta}{\delta J(x_1)}\left[-\int d^4y \; D_F(x_2 - y)J(y)\right] Z[J, J^*]\bigg|_{J,J^*=0}$$
$$= D_F(x_2 - x_1) \quad .$$

This gives the momentum space propagator

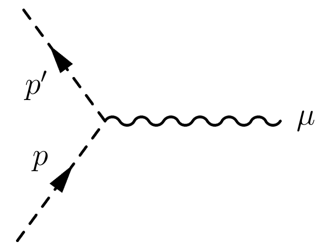


$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

We now look to the interaction terms and write

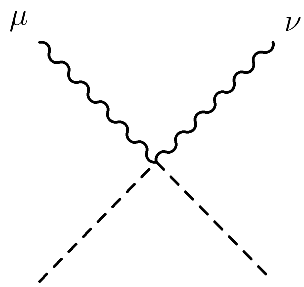
$$\exp\left[i\int d^4x \; \mathcal{L}\right] = \exp\left[i\int d^4x \; \mathcal{L}_0\right]\left(1 + e\int d^4x \; A_\mu\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)\right.$$
$$\left. + ie^2 g^{\mu\nu}\int d^4x \; A_\mu A_\nu \phi^*\phi + \ldots\right) \quad .$$

In momentum space the second term gives the rule



$$= -ie\,(p + p')^{\mu}$$

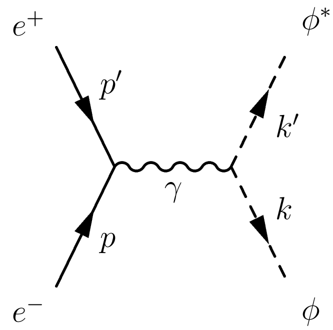
and the third gives



$$= 2ie^{2} g^{\mu\nu} \quad .$$

The factor of 2 comes from the fact that there are two ways that the A_μ and A_ν can be contracted with the external photon states.

(b) The diagram for lowest order $e^+e^- \to \phi\phi^*$ scattering is:



Using the rules from part (a) we can immediately write down the amplitude:

$$\begin{aligned} i\mathcal{M} &= \bar{v}(p')\gamma_\mu u(p) \frac{-ie}{(p+p')^2} \left[-ie(k-k')^{\mu}\right] \\ &= \frac{-e^2(k-k')^{\mu}}{(p+p')^2} \bar{v}(p')\gamma_\mu u(p) \quad . \end{aligned}$$

Squaring this and averaging over electron spins we find (if we ignore the electron mass)

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{s^2}(k-k')^\mu (k-k')^\nu \left[p_\mu p'_\nu + p_\nu p'_\mu - g_{\mu\nu}(p\cdot p')\right] \\
&= \frac{2e^4}{s^2}\left[(p\cdot k - p\cdot k')(p'\cdot k - p'\cdot k') - (m^2 - k\cdot k')(p\cdot p')\right] \quad .
\end{aligned}
$$

If we move to the center of mass frame:

$$
\begin{aligned}
p &= \frac{E_{\text{CM}}}{2}(1,0,0,1) \\
p' &= \frac{E_{\text{CM}}}{2}(1,0,0,-1) \\
k &= \frac{E_{\text{CM}}}{2}\left(1, \sqrt{1-\frac{4m^2}{E_{\text{CM}}^2}}\sin\theta, 0, \sqrt{1-\frac{4m^2}{E_{\text{CM}}^2}}\cos\theta\right) \\
k' &= \frac{E_{\text{CM}}}{2}\left(1, -\sqrt{1-\frac{4m^2}{E_{\text{CM}}^2}}\sin\theta, 0, -\sqrt{1-\frac{4m^2}{E_{\text{CM}}^2}}\cos\theta\right)
\end{aligned}
$$

then

$$
\begin{aligned}
p\cdot k - p\cdot k' &= -(p'\cdot k - p'\cdot k') \\
&= -\frac{E_{\text{CM}}^2}{2}\sqrt{1-\frac{4m^2}{E_{\text{CM}}^2}}\cos\theta \\
m^2 - k\cdot k' &= -\frac{E_{\text{CM}}^2}{2}\left(1-\frac{4m^2}{E_{\text{CM}}^2}\right) \\
p\cdot p' &= \frac{s}{2} = \frac{E_{\text{CM}}^2}{2} \quad .
\end{aligned}
$$

Plugging these in we find

$$
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{2}\left(1-\frac{4m^2}{E_{\text{CM}}^2}\right)\sin^2\theta \quad .
$$

Referring to equations (A.56) and (A.58) of Peskin and Schroeder we write the differential cross section in the center of mass frame as

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{|\mathbf{p}|}{32\pi^2 E_{\rm CM}^3}\left(\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2\right)\\ &= \frac{1}{64\pi^2 E_{\rm CM}^2}\sqrt{1-\frac{4m^2}{E_{\rm CM}^2}}\left[\frac{e^4}{2}\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)\sin^2\theta\right]\\ &= \frac{e^4\sin^2\theta}{128\pi^2 E_{\rm CM}^2}\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)^{3/2}\\ &= \frac{\alpha^2\sin^2\theta}{8E_{\rm CM}^2}\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)^{3/2} \quad .\end{aligned}$$

This gives

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2\sin^2\theta}{4E_{\rm CM}^2}\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)^{3/2}$$

or

$$\sigma = \frac{\pi\alpha^2}{3E_{\rm CM}^2}\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)^{3/2} \quad .$$

Additional Problem

We write down the second order correction (Eq. (6.38)) to the vertex function in the Landau gauge:

$$\delta\Gamma^\mu(p',p) = -ie^2\int\frac{d^4k}{(2\pi)^4}\left(g_{\nu\rho}-\frac{(k-p)_\nu(k-p)_\rho}{(k-p)^2}\right)\frac{\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^\rho}{(k-p)^2(k'^2-m^2)(k^2-m^2)} \quad .$$

We expect a logarithmic divergence since

$$\delta\Gamma^\mu(p',p) \propto \int d^4k\ k^{-4}\left[1+\mathcal{O}(k^{-1})\right] \quad .$$

However, we will show that the coefficient of the k^{-4} term is zero so that

$$\delta\Gamma^\mu(p',p) \propto \int d^4k\ k^{-6}\left[1+\mathcal{O}(k^{-2})\right]$$

and thus is finite in the UV. (Note that we will not obtain a term $\propto k^{-5}$ since our integral is symmetric in k so only even powers will contribute.) We pull out the leading behavior of the integrand:

$$\begin{aligned}\delta\Gamma^\mu(p',p) &= -ie^2\int\frac{d^4k}{(2\pi)^4}\left\{\left(g_{\nu\rho}-\frac{k_\nu k_\rho}{k^2}\right)\frac{\gamma^\nu \not{k}\gamma^\mu \not{k}\gamma^\rho}{k^6}+\mathcal{O}(k^{-5})\right\}\\ &= -ie^2\int\frac{d^4k}{(2\pi)^4}\left\{\frac{-2k_\alpha k_\beta\gamma^\alpha\gamma^\mu\gamma^\beta-k^2\gamma^\mu}{k^6}+\mathcal{O}(k^{-5})\right\}\end{aligned}$$

When evaluating this integral, only even powers of k in the numerator contribute. Therefore we can make the replacement $4k_\alpha k_\beta \to k^2 g_{\alpha\beta}$. We also see that the order k^{-5} term will not contribute as mentioned above. Then we find

$$\begin{aligned}\delta\Gamma^\mu(p',p) &= -ie^2\int\frac{d^4k}{(2\pi)^4}\left\{\frac{-\frac{1}{2}\gamma^\alpha\gamma^\mu\gamma_\alpha-\gamma^\mu}{k^4}+\mathcal{O}(k^{-6})\right\}\\ &= -ie^2\int\frac{d^4k}{(2\pi)^4}\left\{\frac{\gamma^\mu-\gamma^\mu}{k^4}+\mathcal{O}(k^{-6})\right\}\\ &= -ie^2\int\frac{d^4k}{(2\pi)^4}\mathcal{O}(k^{-6})\end{aligned}$$

which is finite.