QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEMS DUE TUESDAY, JANUARY 17

Additional problem: The imaginary part of $\hat{\Pi}_2$ is given in Eq. (7.92) of the text:

$$\mathrm{Im}[\hat{\Pi}_2(q^2 \pm i\epsilon)] = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right) \quad . \tag{1}$$

As discussed in class, one may reconstruct $\hat{\Pi}_2(s)$ from this quantity using a once-subtracted dispersion relation:

$$\hat{\Pi}_2(s) = \Pi_2(s) - \Pi_2(0) = \frac{s}{\pi}\int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}[\hat{\Pi}_2(s'+i\epsilon)]}{s'(s'-s)} \quad . \tag{2}$$

Show that this is equivalent to the expression (7.91) in terms of Feynman parameters:

$$\hat{\Pi}_2(s) = \frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\log\left(\frac{m^2 - x(1-x)s}{m^2}\right) \quad . \tag{3}$$

Hints: (1) Get rid of the log in (3) by integrating by parts. (2) Symmetrize the integrand in the resulting integral around $x = 1/2$ so that you only have to integrate from 0 to 1/2 in x. (3) Make the change of variables $m^2/s' = x(1-x)$.

Solutions — Problem Set 2

David McKeen – January 10, 2006

Problem 7–1

We have the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad .$$

From this we can write down the amplitude for the s-channel loop diagram:

$$\begin{aligned} i\mathcal{M}_s &= \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(k/2-q)^2 - m^2 + i\epsilon}\frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \\ &= \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2 - \Delta)^2} \end{aligned}$$

with $\ell \equiv q + k(1-2x)/2$ and $\Delta \equiv -k^2x(1-x) + m^2$. We use formula (A.44) of Peskin and Schroeder to carry out this integration in d dimensions:

$$\begin{aligned} i\mathcal{M}_s &= \frac{\lambda^2}{2}\int_0^1 dx\ \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-d/2} \\ &= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\ \Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2} \\ &= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\ \left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi}{\Delta}\right)\right) \quad . \end{aligned}$$

Now,

$$\begin{aligned} \int_0^1 dx \log\Delta &= \int_0^1 dx \log[m^2 - sx(1-x)] \\ &= \log m^2 - \int_0^1 dx \left(\frac{2sx^2 - sx}{sx^2 - sx + m^2}\right) \\ &= \log m^2 - 2 + \frac{1}{2}\left(4m^2 - s\right)\int_0^1 \frac{dx}{sx^2 - sx + m^2} \\ &= \log m^2 - 2 + 2\left(\sqrt{\frac{4m^2}{s}} - 1\right)\tan^{-1}\left(\frac{1}{\sqrt{4m^2/s - 1}}\right) \\ &= \log m^2 - 2 + 2\left(\sqrt{\frac{4m^2}{s}} - 1\right)\sin^{-1}\left(\frac{\sqrt{s}}{2m}\right) \quad . \end{aligned}$$

(Gradshteyn and Ryzhik, 5th Edition, Formulae 2.172 and 2.175 were helpful here.) Using this we find that

$$\mathcal{M}_s = \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi}{m^2}\right) + 2 - 2\left(\sqrt{\frac{4m^2}{s} - 1}\right)\sin^{-1}\left(\frac{\sqrt{s}}{2m}\right)\right) \quad .$$

We would like to analytically continue this to $s > 4m^2$. We see that square roots pick up factors of i in that case:

$$\sqrt{\frac{4m^2}{s} - 1} \to i\sqrt{1 - \frac{4m^2}{s}} \quad .$$

Also, we recall that

$$\sin^{-1} z = -i\log(iz + \sqrt{1 - z^2})$$

so that continuing to $z > 1$ we find

$$\begin{aligned}
\sin^{-1} z &\to -i\log(iz + i\sqrt{z^2 - 1}) \\
&= -\frac{\pi}{2} - i\log(z + \sqrt{z^2 - 1}) \\
&= -\frac{\pi}{2} - i\cosh^{-1} z \quad .
\end{aligned}$$

(Note: we get $-\pi/2$ instead of $\pi/2$ because we define our arguments relative to the negative real axis.) Therefore

$$\begin{aligned}
\sqrt{\frac{4m^2}{s} - 1}\sin^{-1}\left(\frac{\sqrt{s}}{2m}\right) &\to i\sqrt{1 - \frac{4m^2}{s}}\left(-\frac{\pi}{2} - i\cosh^{-1}\left(\frac{\sqrt{s}}{2m}\right)\right) \\
&= \sqrt{1 - \frac{4m^2}{s}}\left(\cosh^{-1}\left(\frac{\sqrt{s}}{2m}\right) - i\frac{\pi}{2}\right) \quad .
\end{aligned}$$

Using this we can continue our matrix element to $s > 4m^2$:

$$\text{Re } \mathcal{M}_s = \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi}{m^2}\right) + 2 - 2\left(\sqrt{1 - \frac{4m^2}{s}}\right)\cosh^{-1}\left(\frac{\sqrt{s}}{2m}\right)\right)$$

and

$$\text{Im } \mathcal{M}_s = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{s}} \quad .$$

By crossing symmetry, we can find the amplitude for the t- and u-channel by simply substituting them for s. In the case of forward scattering $t = 0$. We then get

$$\mathcal{M}_t = \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi}{m^2}\right)\right)$$

. Above threshold $s > 4m^2$ so $u = 4m^2 - s - t < 0$. Then

$$\mathcal{M}_u = \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi}{m^2}\right) + 2 - 2\left(\sqrt{1+\frac{4m^2}{u}}\right)\sinh^{-1}\left(\frac{\sqrt{-u}}{2m}\right)\right) \quad .$$

We see that both $\mathcal{M}_t$ and $\mathcal{M}_u$ are real. Therefore the only contribution to the imaginary part of the amplitude comes from the s-channel. We use the optical theorem to find the total cross section ($s = E_{CM}^2$):

$$\begin{aligned}
\sigma_{tot} &= (2p_{CM}E_{CM})^{-1}\text{Im }\mathcal{M} \\
&= \frac{\lambda^2}{64\pi E_{CM}^2}\frac{\sqrt{E_{CM}^2 - 4m^2}}{\sqrt{E_{CM}^2/4 - m^2}} \\
&= \frac{\lambda^2}{32\pi E_{CM}^2}
\end{aligned}$$

as found in Eq. (4.100) of Peskin and Schroeder and so, the optical theorem is demonstrated to λ^2.

Additional Problem

We start with the expression

$$\hat{\Pi}_2(s) = \frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)\log\left(\frac{m^2 - x(1-x)s}{m^2}\right)$$

and symmetrize it about $x = 1/2$:

$$\begin{aligned}
\hat{\Pi}_2(s) &= \frac{2\alpha}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\ \left(\frac{1}{4} - x^2\right)\log\left(\frac{m^2 - (1/4 - x^2)s}{m^2}\right) \\
&= \frac{4\alpha}{\pi}\int_0^{\frac{1}{2}} dx\ \left(\frac{1}{4} - x^2\right)\log\left(\frac{m^2 - (1/4 - x^2)s}{m^2}\right) \quad .
\end{aligned}$$

We integrate by parts obtaining

$$\begin{aligned}
\hat{\Pi}_2(s) &= -\frac{4\alpha}{\pi}\int_0^{\frac{1}{2}} dx\ \left(\frac{x}{4} - \frac{x^3}{3}\right)\left(\frac{m^2}{m^2 - (1/4 - x^2)s}\right)\left(\frac{2xs}{m^2}\right) \\
&= \frac{4\alpha s}{\pi}\int_0^{\frac{1}{2}}\left(\frac{2x dx}{m^2}\right)\frac{x}{3}\left(x^2 - \frac{3}{4}\right)\left(\frac{m^2}{m^2 - (1/4 - x^2)s}\right) \quad .
\end{aligned}$$

Now, we make a change of variables: $m^2/s' = 1/4 - x^2$ so that s' runs from $4m^2$ to ∞ and $2x dx/m^2 = ds'/(s')^2$. Then

$$\begin{aligned}
\hat{\Pi}_2(s) &= \frac{1}{3}\frac{4\alpha s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{(s')^2}\left(\frac{1}{2}\right)\sqrt{1-\frac{4m^2}{s'}}\left(-\frac{1}{2}\right)\left(1+\frac{2m^2}{s'}\right)\left(\frac{1}{1-s/s'}\right) \\
&= -\frac{\alpha s}{3\pi}\int_{4m^2}^{\infty} ds' \frac{\sqrt{1-4m^2/s'}\,(1+2m^2/s')}{s'(s'-s)} \\
&= \frac{s}{\pi}\int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\left[\hat{\Pi}_2(s'+i\epsilon)\right]}{s'(s'-s)}
\end{aligned}$$

with

$$\mathrm{Im}\left[\hat{\Pi}_2(s' \pm i\epsilon)\right] = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{s'}}\left(1+\frac{2m^2}{s'}\right)$$