QUANTUM FIELD THEORY I

Physics 444 - Winter Quarter, 2006 - University of Chicago

PROBLEMS DUE TUESDAY, JANUARY 10

Problem in text	**Subject**
6–1	Rosenbluth Formula

Hint: The trace algebra is considerably simpler if you use the Gordon identity in Problem 3.2 to substitute for the $i\sigma^{\mu\nu}q_\nu/2m$ term. You will also have to derive the phase space formula for the laboratory frame, repeating the steps we took for the center-of-mass frame.

Solutions — Problem Set 1

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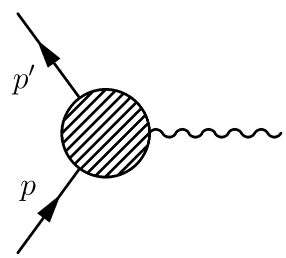
Problem 6–1

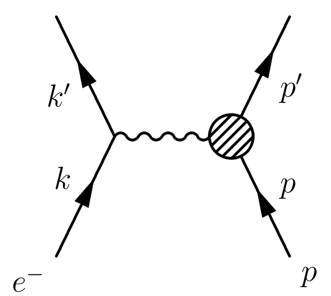
Lorentz invariance restricts the form of the Feynman rule for the proton-photon vertex:



$$= ie\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]$$

$$= ie\left[\gamma^\mu\left(F_1 + F_2\right) - \frac{p^\mu + p'^\mu}{2m}F_2\right]$$

where $F_1(q^2)$ and $F_2(q^2)$ are determined by QCD and $q = p' - p$. We consider the scattering of an electron off a proton initially at rest:



.

Then, to lowest order in α, the amplitude is given by

$$
\begin{aligned}
i\mathcal{M} &= \{\bar{u}_e(k')\,(-ie\gamma_\mu)\,u_e(k)\}\left(\frac{-i}{q^2}\right)\left\{\bar{u}_p(p')\left[ie\left(\gamma^\mu\,(F_1+F_2)-\frac{p^\mu+p'^\mu}{2m}F_2\right)\right]u_p(p)\right\} \\
&= -\frac{ie^2}{q^2}\bar{u}_e(k')\gamma_\mu u_e(k)\bar{u}_p(p')\left[\gamma^\mu\,(F_1+F_2)-\frac{p^\mu+p'^\mu}{2m}F_2\right]u_p(p)\ .
\end{aligned}
$$

Squaring this, summing over final spins, and averaging over initial spins we find

$$
\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{e^4}{4q^4}\mathrm{Tr}(\not{k}'\gamma_\mu\,\not{k}\gamma_\nu)\mathrm{Tr}\left[(\not{p}'+m)\,(A\gamma^\mu-B^\mu)\,(\not{p}+m)\,(A\gamma^\nu-B^\nu)\right]
$$

with $A\equiv F_1+F_2$ and $B^\mu\equiv(p^\mu+p'^\mu)F_2/2m$. The first trace is

$$
\mathrm{Tr}(\not{k}'\gamma_\mu\,\not{k}\gamma_\nu) = 4\left[k_\mu k_{\prime\nu}+k_\nu k_{\prime\mu}-g_{\mu\nu}(k\cdot k')\right]\ .
$$

The second is

$$
\begin{aligned}
\mathrm{Tr}\left[(\not{p}'+m)\,(A\gamma^\mu-B^\mu)\,(\not{p}+m)\,(A\gamma^\nu-B^\nu)\right] &= A^2\mathrm{Tr}(\not{p}'\gamma^\mu\,\not{p}\gamma^\nu)-mAB^\nu\mathrm{Tr}(\not{p}'\gamma^\mu) \\
&\quad + B^\mu B^\nu\mathrm{Tr}(\not{p}'\,\not{p})-mAB^\mu\mathrm{Tr}(\not{p}'\gamma^\nu)-mAB^\mu\mathrm{Tr}(\not{p}\gamma^\nu)+m^2B^\mu B^\nu\mathrm{Tr}(\mathbf{1}) \\
&\quad - mAB^\nu\mathrm{Tr}(\gamma^\mu\,\not{p})+m^2A^2\mathrm{Tr}(\gamma^\mu\gamma^\nu) \\
&= 4A^2\left[p^\mu p'^\nu+p^\nu p'^\mu-g^{\mu\nu}(p\cdot p')\right]-4mAB^\nu p'^\mu+4B^\mu B^\nu(p\cdot p')-4mAB^\mu p'^\nu \\
&\quad - 4mAB^\mu p^\nu+4m^2B^\mu B^\nu-4mAB^\nu p^\mu+4g^{\mu\nu}m^2A^2 \\
&= 4A^2\left[p^\mu p'^\nu+p^\nu p'^\mu-g^{\mu\nu}(p\cdot p'-m^2)\right]+4B^\mu B^\nu(p\cdot p'+m^2) \\
&\quad - 4mAB^\mu(p^\nu+p'^\nu)-4mAB^\nu(p^\mu+p'^\mu) \\
&= 4\Big\{(F_1+F_2)^2\left[p^\mu p'^\nu+p^\nu p'^\mu-g^{\mu\nu}(p\cdot p'-m^2)\right] \\
&\quad + \left[\frac{F_2^2}{4m^2}(p\cdot p'+m^2)-(F_1+F_2)F_2\right](p^\mu+p'^\mu)(p^\nu+p'^\nu)\Big\} \\
&= 4(F_1+F_2)^2\left[p^\mu p'^\nu+p^\nu p'^\mu-g^{\mu\nu}(p\cdot p'-m^2)\right] \\
&\quad + \left[-F_2^2(2+\frac{q^2}{2m^2})-4F_1F_2\right](p^\mu+p'^\mu)(p^\nu+p'^\nu)\Big\}
\end{aligned}
$$

At this point we thank Peskin and Schroeder for including the answer. We manipulate the second term to obtain a factor of $F_1^2 - q^2F_2^2/4m^2$ by adding and subtracting $2F_1^2$. We find

$$\begin{aligned}\mathrm{Tr}\left[(\not{p}' + m)\left(A\gamma^\mu - B^\mu\right)(\not{p} + m)\left(A\gamma^\nu - B^\nu\right)\right] &= 4(F_1 + F_2)^2\left[p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p' - m^2)\right] \\ &+ \left[2(F_1^2 - \frac{q^2}{4m^2}F_2^2) - 2(F_1 + F_2)^2\right](p^\mu + p'^\mu)(p^\nu + p'^\nu)\ .\end{aligned}$$

Now,

$$2\left[p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p' - m^2)\right] - (p^\mu + p'^\mu)(p^\nu + p'^\nu) = q^2 g^{\mu\nu} - q^\mu q^\nu$$

(using $q^2 = 2m^2 - 2p\cdot p'$). Then

$$\begin{aligned}\mathrm{Tr}\left[(\not{p}' + m)\left(A\gamma^\mu - B^\mu\right)(\not{p} + m)\left(A\gamma^\nu - B^\nu\right)\right] &= 2(F_1 + F_2)^2(q^2 g^{\mu\nu} - q^\mu q^\nu) \\ &+ 2(F_1^2 - \frac{q^2}{4m^2}F_2^2)(p^\mu + p'^\mu)(p^\nu + p'^\nu)\ .\end{aligned}$$

This gives us

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{2e^4}{q^4}\left[C(F_1 + F_2)^2 + D\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\right]$$

with

$$\begin{aligned}C &= \left[k_\mu k_{\prime\nu} + k_\nu k_{\prime\mu} - g_{\mu\nu}(k\cdot k')\right](q^2 g^{\mu\nu} - q^\mu q^\nu) \\ &= -q^2(k\cdot k') - 2(q\cdot k)(q\cdot k') \\ &= (k\cdot k')\left[2(k\cdot k') - q^2\right] \\ &= 4(k\cdot k')^2\end{aligned}$$

and

$$\begin{aligned}D &= \left[k_\mu k_{\prime\nu} + k_\nu k_{\prime\mu} - g_{\mu\nu}(k\cdot k')\right](p^\mu + p'^\mu)(p^\nu + p'^\nu) \\ &= 2(p\cdot k + p'\cdot k)(p\cdot k' + p'\cdot k') - (p + p')^2(k\cdot k')\ .\end{aligned}$$

We need to do some kinematics now. We will use Mandelstam variables since they clean things up. They are

$$
\begin{aligned}
s &= (p+k)^2 = m^2 + 2p \cdot k \\
&= (p'+k')^2 = m^2 + 2p' \cdot k' \\
t &= (p-p')^2 = 2m^2 - 2p \cdot p' \\
&= (k'-k)^2 = -2k \cdot k' \\
&= q^2 \\
u &= (p-k')^2 = m^2 - 2p \cdot k' \\
&= (k-p')^2 = m^2 - 2p' \cdot k \ .
\end{aligned}
$$

We can then write

$$
C = t^2
$$

and

$$
\begin{aligned}
D &= \frac{1}{2}(s-u)^2 + \frac{1}{2}t(4m^2 - t) \\
&= \frac{1}{2}(2s + t - 2m^2)^2 + \frac{1}{2}t(4m^2 - t) \\
&= 2(s-m^2)^2 + 2st \ .
\end{aligned}
$$

Now, we specialize to the lab frame. We take

$$
\begin{aligned}
p &= (m, 0) \\
k &= (E, E\hat{z}) \\
p' &= (E'_p, \mathbf{p}'_p) \\
k' &= (E', E' \sin\theta \hat{x} + E' \cos\theta \hat{z}) \ .
\end{aligned}
$$

We can express the photon's final energy by use of

$$m^2 = (p')^2 = (p + k - k')^2 = m^2 + 2p \cdot (k - k') - 2k \cdot k'$$
$$= m^2 + 2m(E - E') - 2EE'(1 - \cos\theta)$$

which gives

$$E' = \frac{E}{1 + \frac{E}{m}(1 - \cos\theta)} = \frac{E}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} \ .$$

The Mandelstam variables in the lab frame become

$$s = m^2 + 2mE$$
$$t = -2EE'(1 - \cos\theta)$$
$$= -\frac{4E^2 \sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} \ .$$

We use these to write

$$C = -\frac{4q^2E^2\sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} = -4q^2EE'\sin^2\frac{\theta}{2}$$

and

$$D = 2(2mE)^2 + 2(m^2 + 2mE)\left(-\frac{4E^2\sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}}\right)$$
$$= \frac{8m^2E^2\cos^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} = 8m^2EE'\cos^2\frac{\theta}{2} \ .$$

Plugging this in we get the invariant amplitude in the lab frame:

$$\left(\frac{1}{4}\sum|\mathcal{M}|^2\right)_{\text{lab}} = \frac{16e^4m^2EE'}{\left(-4EE'\sin^2\frac{\theta}{2}\right)^2}\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1 + F_2)^2\sin^2\frac{\theta}{2}\right]$$
$$= \frac{e^4m^2}{EE'\sin^4\frac{\theta}{2}}\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1 + F_2)^2\sin^2\frac{\theta}{2}\right] \ .$$

In the lab frame, the 2-body phase space integral is

$$\begin{aligned}
\int d\Pi_2 &= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2E'}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E'_p}(2\pi)^4\delta^{(0)}(p'+k'-p-k) \\
&= \frac{1}{16\pi^2}\int dE'd\Omega\frac{E'}{E'_p} \\
&\qquad \times \delta(\sqrt{m^2+E^2+E'^2-2EE'\cos\theta}+E'-m-E) \\
&= \frac{1}{8\pi}\int d\cos\theta\frac{E'}{E'_p}\frac{1}{\left|\frac{E'-E\cos\theta}{E'_p}+1\right|} \\
&= \frac{1}{8\pi}\int d\cos\theta\frac{E'}{m+E(1-\cos\theta)} \\
&= \frac{1}{8\pi}\int d\cos\theta\frac{E'^2}{mE}
\end{aligned}$$

The formula connecting the amplitude to the cross section is

$$d\sigma = \frac{1}{4E_eE_p\left|v_e-v_p\right|}\int d\Pi_2\left|\mathcal{M}\right|^2$$

Using $v_e \approx 1$, $v_p \approx 0$, $E_e = E$, and $E_p = m$ we get

$$\begin{aligned}
\left(\frac{d\sigma}{d\cos\theta}\right)_{\text{lab}} &= \frac{E'^2}{32\pi m^2E^2}\left(\frac{1}{4}\sum\left|\mathcal{M}\right|^2\right)_{\text{lab}} \\
&= \frac{e^4}{32\pi E^2\sin^4\frac{\theta}{2}}\left(\frac{E'}{E}\right)\left[(F_1^2-\frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right] \\
&= \frac{\pi\alpha^2\left[(F_1^2-\frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left[1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}}\ .
\end{aligned}$$