QUANTUM FIELD THEORY II

Physics 444 - Winter Quarter, 2006 - University of Chicago

FINAL PROBLEM SET DUE TUESDAY, March 14

(1) Write a one-dimensional integral representation (either in terms of a Feynman parameter x or a dispersion variable s') for the photon vacuum polarization tensor $\Pi_{\mu\nu}(q)$ due to a scalar field with unit charge. Using this result, write the corresponding contribution to the beta function.

(2) One way to calculate the beta function for a massless vector field is to use (without proof) the property that for N=4 supersymmetric SU(M) the beta function vanishes. An N=4 supermultiplet contains the following field content:

$J_z = 1$: 1 field
$J_z = 1/2$: 4 fields
$J_z = 0$: 6 fields
$J_z = -1/2$: 4 fields
$J_z = -1$: 1 field

You already know how to calculate the beta functions for fermions ($J_z = \pm 1/2$) and scalars ($J_z = 0$, using part (1)). Using the fact that the sum should vanish for the above supermultiplet, calculate the beta function for the vector meson ($J_z = 1$ or -1) belonging to the adjoint representation of SU(M).

Solutions — Final Problem Set

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Problem 1 There are two contributions to the photon vacuum polarization tensor $\Pi^{\mu\nu}$ from charged scalar fields, arising respectively from the photon-scalar-scalar vertex to second order ($\Pi_1^{\mu\nu}$) and the photon-photon-scalar-scalar vertex to first order ($\Pi_2^{\mu\nu}$). Both contributions are needed for gauge invariance.

The first contribution may be written

$$i\Pi_1^{\mu\nu} = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\mu (2k-q)^\nu}{(k^2-m^2)[(k-q)^2-m^2]} (i)^2 \tag{1}$$

while the second may be written

$$i\Pi_2^{\mu\nu} = 2ie^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \quad . \tag{2}$$

It is best to combine these two terms by multiplying the numerator and denominator of the second integrand by $(k-q)^2 - m^2$. Adopting a Feynman parametrization in which

$$\frac{1}{(k^2-m^2)[(k-q)^2-m^2]} = \int_0^1 dx \frac{1}{D^2} \ , \quad D \equiv x[(k-q)^2-m^2] + (1-x)(k^2-m^2) = (k-xq)^2 - \Delta \ , \tag{3}$$

with $\Delta \equiv m^2 - x(1-x)q^2$, and defining the shifted variable $\ell \equiv k - xq$, one can write the combined terms as

$$i\Pi^{\mu\nu} = i\Pi_1^{\mu\nu} + i\Pi_2^{\mu\nu} = e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{N^{\mu\nu}}{(\ell^2-\Delta)^2} \quad ,$$

$$N^{\mu\nu} \equiv [2\ell + (2x-1)q]^\mu \ [2\ell + (2x-1)q]^\nu - 2g^{\mu\nu}[(\ell + (x-1)q)^2 - m^2] \quad . \tag{4}$$

Equivalently, one can rewrite the numerator function as

$$N^{\mu\nu} = \left\{ \left[\frac{4}{d} - 2 \right] \ell^2 - 2[(x-1)^2 q^2 - m^2] \right\} g^{\mu\nu} + (2x-1)^2 q^\mu q^\nu \quad , \tag{5}$$

where terms linear in ℓ have been dropped and one has substituted in the usual way $\ell^\mu \ell^\nu = (1/d) g^{\mu\nu} \ell^2$. Performing the ℓ integration after rotating to Euclidean momenta, one finds

$$\begin{aligned} \Pi^{\mu\nu} &= e^2 \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \left[(d-2)\Gamma(1-\frac{d}{2}) g^{\mu\nu} \Delta \right. \\ &+ \left. \{2[m^2 - q^2(1-x)^2] g^{\mu\nu} + (2x-1)^2 q^\mu q^\nu\} \Gamma(2-\frac{d}{2}) \right] \quad . \end{aligned} \tag{6}$$

Since $(d-2)\Gamma(1-\frac{d}{2}) = -2\Gamma(2-\frac{d}{2})$, the coefficient of $g^{\mu\nu}$ in the large square brackets becomes $2[-\Delta + m^2 - q^2(1-x)^2] = -2q^2(1-x)(1-2x)$. Now, $1-2x$ is odd under the substitution $x \leftrightarrow 1-x$, so one may decompose its coefficient $1-x = \frac{1}{2}(1-2x) + \frac{1}{2}$ and keep only the $1-2x$ term, since the remaining terms in the integral are even under $x \leftrightarrow 1-x$. Thus one finds that the coefficient of $q^2 g^{\mu\nu}$ and $q^\mu q^\nu$ are equal and opposite, and

$$\Pi^{\mu\nu} = -e^2 \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} (2x-1)^2 (q^2 g^{\mu\nu} - q^\mu q^\nu) \quad . \tag{7}$$

If we were to renormalize this quantity so that it vanished at $q^2 = 0$, we would have

$$\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{\alpha}{4\pi} \int_0^1 dx (2x-1)^2 \log \frac{m^2 - x(1-x)q^2}{m^2} \quad . \tag{8}$$

Renormalizing it instead at $q^2 = -M^2$ we would have

$$\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{\alpha}{4\pi} \int_0^1 dx (2x-1)^2 \log \frac{m^2 - x(1-x)q^2}{m^2 + x(1-x)M^2} \quad . \tag{9}$$

With $\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ we see that for $q^2 \to \infty$,

$$\Pi(q^2) \to \frac{\alpha}{12\pi} \log(-q^2/M^2) \quad , \tag{10}$$

or 1/4 of the corresponding asymptotic value for a spin-1/2 fermion of unit charge.

An alternative expression is based on the dispersion integral for $\Pi(q^2)$. The imaginary part of $\Pi(q^2)$ is just given by an expression similar to the imaginary part for fermions, but with a slightly different coefficient and kinematic factor:

$$\text{Scalar}: \ \text{Im}[\Pi(q^2 \pm i\epsilon)] = \mp \frac{\alpha}{12} \left(1 - \frac{4m^2}{q^2}\right)^{3/2} \quad , \ \text{vs.} \tag{11}$$

$$\text{Fermion}: \ \text{Im}[\Pi(q^2 \pm i\epsilon)] = \mp \frac{\alpha}{3} \left(1 + \frac{2m^2}{q^2}\right) \left(1 - \frac{4m^2}{q^2}\right)^{1/2} \quad . \tag{12}$$

We write a once-subtracted dispersion relation for $\Pi(q^2)$ [assuming that $\Pi(0) = 0$]:

$$\Pi(s) = \frac{s}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im}[\Pi(s' + i\epsilon)]}{s'(s'-s)} \quad , \tag{13}$$

which is an acceptable representation if the imaginary part is given as above. However, it can be transformed into the form given by an integral over the Feynman parameter x. First one defines

$m^2/s' = \frac{1}{4} - z^2$ and finds

$$\Pi(s) = -\frac{4s\alpha}{3\pi}\int_0^{1/2}\frac{z^4 dz}{m^2}\frac{m^2}{m^2-(\frac{1}{4}-z^2)s} \quad . \tag{14}$$

This may be integrated by parts to cast it into the form containing a log, since

$$\frac{d}{dz}\log\left(\frac{m^2-(\frac{1}{4}-z^2)s}{m^2}\right) = \frac{2zs}{m^2}\frac{m^2}{m^2-(\frac{1}{4}-z^2)s} \quad . \tag{15}$$

Then one finds

$$\begin{aligned}\Pi(s) &= -\frac{2\alpha}{3\pi}\int_0^{1/2} x^3 dz \log\left(\frac{m^2-(\frac{1}{4}-z^2)s}{m^2}\right) \\ &= \frac{\alpha}{\pi}\int_{-1/2}^{1/2} z^2 dz \log\left(\frac{m^2-(\frac{1}{4}-z^2)s}{m^2}\right) \quad .\end{aligned} \tag{16}$$

Now let $x = z + 1/2$, so that $x(1-x) = \frac{1}{4} - z^2$, $z^2 = (x - \frac{1}{2})^2$, and one has

$$\Pi(s) = \frac{\alpha}{4\pi}\int_0^1 (2x-1)^2 dx \log\left(\frac{m^2 - x(1-x)s}{m^2}\right) \tag{17}$$

in agreement with the result (8) obtained in terms of a Feynman parameter. One could renormalize instead at $q^2 = -M^2$ and one would then obtain the result (9). The asymptotic form of $\Pi(s)$ as $s \to \infty$ is easily seen from its representation (13) in terms of a dispersion integral and the asymptotic behavior of $\mathrm{Im}[\Pi(s')]$ in Eq. (11).

Comparing the asymptotic forms of (11) and (12) as $q^2 \to -\infty$ (which are all that are needed for the beta functions) one sees immediately that the scalar beta-function is 1/4 that of a fermion with the same charge.

Problem 2 We take as given the fact that the beta function vanishes for the full $N = 4$ supermultiplet, which consists of one massless vector field with helicities $\lambda = \pm 1$, four massless spinor fields with helicities $\lambda = \pm 1/2$, and six scalar fields with helicity $\lambda = 0$, all belonging to the *adjoint* representation of SU(M). Thus the beta function β_1 for the vector field, the beta function $\beta_{1/2}$ for the spinor field, and the beta function β_0 for the scalar field must satisfy the relation

$$\beta_1 + 4\beta_{1/2} + 6\beta_0 = 0 \quad . \tag{18}$$

Now, we can repeat the calculation of $\beta_{1/2}$ done earlier for fermions belonging to the fundamental M-dimensional representation of SU(M), for which $C(r) = 1/2$, but now with fermions in the gauge boson loop diagram belonging to the adjoint representation. Thus, for the representation matrix t_r at each end of the loop we have i times a structure constant, so $\mathrm{Tr}(t_r^a t_r^b) = C(r)\delta^{ab}$ becomes $if^{cad}if^{dbc} = C_2(G)\delta^{ab} = M\delta^{ab}$. Then we have

$$\beta_{1/2} = \frac{g^3}{(4\pi)^2}\frac{2}{3}C_2(G) \quad . \tag{19}$$

Moreover we have shown that $\beta_0 = \frac{1}{4}\beta_{1/2}$. Then

$$\beta_1 = -4\beta_{1/2} - \frac{6}{4}\beta_{1/2} = -\frac{11}{2}\beta_{1/2} = -\frac{g^3}{(4\pi)^2}\frac{11}{3}C_2(G) \quad . \tag{20}$$