

QUANTUM FIELD THEORY I

Physics 443 - Fall Quarter, 2005 - University of Chicago

PROBLEMS DUE TUESDAY, NOVEMBER 29

Problem in text	Subject
5-4(a)	Positronium Lifetime

Solutions — Problem Set 6

David McKeen – November 22, 2005

Problem 5–2

The process $e^-e^+ \rightarrow \gamma\gamma$ involves the t - and u -channel diagrams:

$$\begin{aligned}
 i\mathcal{M} &= \text{[t-channel diagram]} + \text{[u-channel diagram]} \\
 &= -e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \bar{v}(p_2) \left\{ \gamma^\nu \frac{\not{p}_1 - \not{k}_1 + m}{(p_1 - k_1)^2 - m^2} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2 + m}{(p_1 - k_2)^2 - m^2} \gamma^\nu \right\} u(p_1) .
 \end{aligned}$$

The two amplitudes constructively interfere since they differ under the exchange of a boson so there no anticommutations. The Dirac equation allows us to write

$$(\not{p}_1 + m)\gamma^\mu u(p_1) = 2p_1^\mu u(p_1)$$

so the amplitude becomes

$$i\mathcal{M} = -e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \bar{v}(p_2) \left\{ \gamma^\nu \frac{2p_1^\mu - \not{k}_1 \gamma^\mu}{(p_1 - k_1)^2 - m^2} + \gamma^\mu \frac{2p_1^\nu - \not{k}_2 \gamma^\nu}{(p_1 - k_2)^2 - m^2} \right\} u(p_1) .$$

In the center of mass frame

$$k_1 = m(1, \hat{n}) \quad k_2 = m(1, -\hat{n})$$

where \hat{n} is a unit 3-vector. In the nonrelativistic limit we can take

$$p_1 \approx p_2 \approx (m, \vec{0}) .$$

Then

$$(p_1 - k_1)^2 \approx (p_1 - k_2)^2 \approx -m^2 .$$

Now, we take the photons to be longitudinally polarized and because of the form of p_1 above, we see that

$$\epsilon_\mu^*(k_1)p_1^\mu \approx \epsilon_\mu^*(k_2)p_1^\mu \approx 0 .$$

The amplitude simplifies to

$$\begin{aligned} i\mathcal{M} &\approx -\frac{e^2}{2m^2} \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \bar{v}(p_2) (\gamma^\nu \not{k}_1 \gamma^\mu + \gamma^\mu \not{k}_2 \gamma^\nu) u(p_1) \\ &= -\frac{e^2}{2m} \epsilon_i^*(k_1) \epsilon_j^*(k_2) \bar{v}(p_2) [\gamma^j (\gamma^0 + n^k \gamma^k) \gamma^i + \gamma^i (\gamma^0 - n^k \gamma^k) \gamma^j] u(p_1) \\ &= -\frac{e^2}{2m} \epsilon_i^*(k_1) \epsilon_j^*(k_2) \bar{v}(p_2) n^k (\gamma^j \gamma^k \gamma^i - \gamma^i \gamma^k \gamma^j) u(p_1) . \end{aligned}$$

We use the Dirac algebra

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij}$$

to see that

$$\begin{aligned} \gamma^j \gamma^k \gamma^i - \gamma^i \gamma^k \gamma^j &= -2\delta^{jk} \gamma^i - \gamma^k \gamma^j \gamma^i + 2\delta^{kj} \gamma^i + \gamma^i \gamma^j \gamma^k \\ &= -(\gamma^k \gamma^j \gamma^i - \gamma^i \gamma^j \gamma^k) . \end{aligned}$$

That is, this object is antisymmetric under interchange of any three of its indices.

Thus it's proportional to ϵ^{ijk} . To get the proportionality constant we set $i = 1$,

$j = 2$, and $k = 3$ and use the chiral basis. We find

$$\begin{aligned}
\gamma^2\gamma^3\gamma^1 - \gamma^1\gamma^3\gamma^2 &= \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma^1\sigma^3\sigma^2 - \sigma^2\sigma^3\sigma^1 \\ -\sigma^1\sigma^3\sigma^2 + \sigma^2\sigma^3\sigma^1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \\
&= -2i\gamma^0\gamma^5
\end{aligned}$$

Thus we can write

$$\gamma^j\gamma^k\gamma^i - \gamma^i\gamma^k\gamma^j = -2i\epsilon^{ijk}\gamma^0\gamma^5$$

So, the amplitude is

$$\begin{aligned}
i\mathcal{M} &\approx \frac{ie^2}{m} \{[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}\} \bar{v}(p_2)\gamma^0\gamma^5u(p_1) \\
&= \frac{ie^2}{m} \{[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}\} v^\dagger(p_2)\gamma^5u(p_1) .
\end{aligned}$$

In the nonrelativistic limit in the chiral representation

$$u(p) \approx \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \quad v(p) \approx \sqrt{m} \begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix} .$$

Then

$$\begin{aligned}
v^\dagger(p_2)\gamma^5u(p_1) &= m \begin{pmatrix} \eta^{s'\dagger} & -\eta^{s'\dagger} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \\
&= -2m \eta^{s'\dagger}\xi^s .
\end{aligned}$$

So

$$i\mathcal{M} \approx -2ie^2 \{[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}\} \eta^{s'\dagger} \xi^s .$$

It suffices to take both spins up to consider the spin 1 case (the other states are equivalent up to a rotation of the coordinate system). The spin up fermion's spinor is

$$\xi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

while the antifermion's is

$$\eta^\uparrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Plugging this in we see that

$$i\mathcal{M}(^3S_1 \rightarrow \gamma\gamma) \propto \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

The singlet state gives a factor of one here. Thus

$$\begin{aligned} \mathcal{M}(^1S_0 \rightarrow \gamma\gamma) &= \sqrt{2(2m)} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{\psi}^*(\mathbf{k}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} \{-2e^2 [\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}\} \\ &= -\frac{2e^2}{\sqrt{m}} \psi^*(0) \{[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}\} \\ &= -\frac{8\pi\alpha}{\sqrt{m}} \psi^*(0) \{[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}\} . \end{aligned}$$

Then, we see that

$$\sum_{\gamma \text{ pol.}} |\mathcal{M}|^2 = \frac{64\pi^2\alpha^2}{m} |\psi(0)|^2 \sum_{\gamma \text{ pol.}} |[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n}|^2 .$$

When summed over photon polarizations

$$[\epsilon^*(k_1) \times \epsilon^*(k_2)] \cdot \hat{n} \rightarrow 2$$

which can be easily seen if we take $\epsilon_1 = \hat{x}$, $\epsilon_2 = \hat{y}$, and $\hat{n} = \hat{z}$. Also, we recall that

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

with $a = (Z\alpha\mu)^{-1}$ and $\mu = m/2$ the reduced mass. Assembling all this we find

$$\begin{aligned} \sum_{\gamma \text{ pol.}} |\mathcal{M}|^2 &= \frac{64\pi^2 \alpha^2}{m} \frac{\alpha^3 m^3}{8\pi} (4) \\ &= 32\pi \alpha^5 m^2 . \end{aligned}$$

The partial decay width is then given by

$$d\Gamma = \frac{1}{2} \frac{1}{2(2m)} \frac{d\Omega}{4\pi} \frac{1}{8\pi} \frac{2|\mathbf{k}|}{E_{cm}} \left(\sum_{\gamma \text{ pol.}} |\mathcal{M}|^2 \right) .$$

Since the matrix element is isotropic, we can easily integrate to get

$$\begin{aligned} \Gamma &= \frac{1}{64\pi m} \left(\sum_{\gamma \text{ pol.}} |\mathcal{M}|^2 \right) \\ &= \frac{\alpha^5 m}{2} \\ &= 5.29 \times 10^{-6} \text{ eV} \\ &= 8.04 \times 10^9 \text{ s}^{-1} \end{aligned}$$