QUANTUM FIELD THEORY I

Physics 443 - Fall Quarter, 2005 - University of Chicago

PROBLEMS DUE TUESDAY, NOVEMBER 22

| **Problem in text** | **Subject** |
|---|---|
| 5–2 | Bhabha scattering |

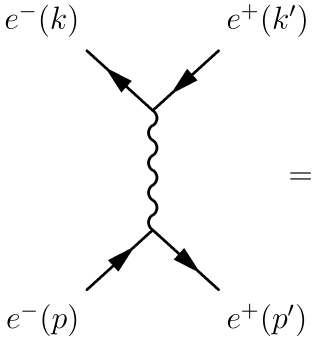
Hint: Assume you have already calculated the differential cross sections for $e^+e^- \rightarrow \mu^+\mu^-$ and $e^-\mu^+ \rightarrow e^-\mu^+$ described in the text in terms of the squared matrix elements, summed over spins, given by Eqs. (5.70) and (5.71) respectively. In Bhabha scattering there are contributions due to both these squared matrix elements, and an additional interference term which requires calculation of one additional trace. You need only calculate this interference term: be sure to get the sign right.

## Solutions — Problem Set 6

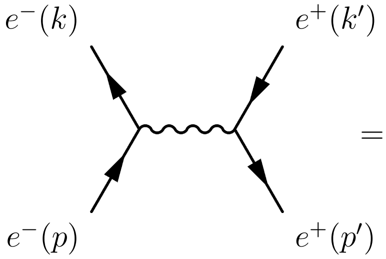
David McKeen – November 22, 2005

# Problem 5–2

We can immediately write down the $s$- and $t$-channel amplitudes:

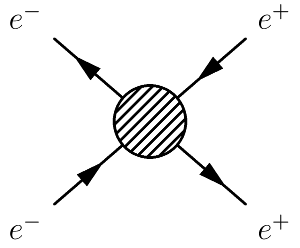


$$= i\mathcal{M}_s = \frac{ie^2}{s}\bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^r(k)\gamma_\mu v^{r'}(k')$$



$$= i\mathcal{M}_t = \frac{ie^2}{t}\bar{u}^r(k)\gamma^\mu u^s(p)\bar{v}^{s'}(p')\gamma_\mu v^{r'}(k').$$

To get the full amplitude we subtract these two. To see this follow the discussion on pp. 119-120 of Peskin and Schroeder and untangle the contractions corresponding to the $s$- and $t$-channel diagrams. You will get relative a factor of $(-1)$ between the two matrix elements because of the anticommutation of Dirac fields. Thus,



$$= i\mathcal{M} = i\mathcal{M}_s - i\mathcal{M}_t$$

We square the amplitude, average over initial spins and sum over final spins. Then

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 + \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_t|^2 - \frac{1}{2}\mathrm{Re}\sum_{\text{spins}} \mathcal{M}_t\mathcal{M}_s^*.$$

Working in the massless limit from here on out, we have

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_s|^2 &= 2e^4\left[\left(\frac{t}{s}\right)^2 + \left(\frac{u}{s}\right)^2\right] \\
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_t|^2 &= 2e^4\left[\left(\frac{s}{t}\right)^2 + \left(\frac{u}{t}\right)^2\right]
\end{aligned}$$

as given by equations (5.70) and (5.71). Now,

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spins}} \mathcal{M}_t\mathcal{M}_s^* &= \frac{e^4}{2st}\sum_{\text{s,s}',\text{r,r}'} \bar{u}^r(k)\gamma^\mu u^s(p)\bar{v}^{s'}(p')\gamma_\mu v^{r'}(k') \\
&\qquad\qquad \times \bar{v}^{r'}(k')\gamma_\nu u^r(k)\bar{u}^s(p)\gamma^\nu v^{s'}(p') \\
&= \frac{e^4}{2st}\mathrm{Tr}\left(\gamma^\mu \not{p}\gamma^\nu \not{p}'\gamma_\mu \not{k}'\gamma_\nu \not{k}\right) \\
&= -\frac{e^4}{st}\mathrm{Tr}\left(\not{p}'\gamma^\nu \not{p}\,\not{k}'\gamma_\nu \not{k}\right) \\
&= -\frac{4e^4}{st}\left(p\cdot k'\right)\mathrm{Tr}\left(\not{p}'\,\not{k}\right) \\
&= -\frac{16e^4}{st}\left(p\cdot k'\right)\left(p'\cdot k\right) \\
&= -4e^4\left(\frac{u^2}{st}\right).
\end{aligned}$$

Using this result we get

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= 2e^4\left[\left(\frac{t}{s}\right)^2+\left(\frac{u}{s}\right)^2+\left(\frac{s}{t}\right)^2+\left(\frac{u}{t}\right)^2+\frac{2u^2}{st}\right]\\ &= 2e^4\left[\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right].\end{aligned}$$

Eqs. (A.56) and (A.58) in the massless limit give us

$$d\sigma = \frac{1}{8E_{e^-}E_{e^+}}\frac{d\Omega}{4\pi}\frac{1}{8\pi}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right)$$

We use the fact that $E_{e^-} = E_{e^+} = \sqrt{s}/2$ in the center of mass frame and integrate over the azimuthal direction (since all of the Mandelstam variables are independent of this direction) to get

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{1}{32\pi s}\left(2e^4\right)\left[\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right]\\ &= \frac{e^4}{16\pi s}\left[\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right].\end{aligned}$$

Using $\alpha = e^2/4\pi$ this gives

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right].$$

In the center of mass frame we can write

$$\begin{aligned}p &= \frac{\sqrt{s}}{2}(1,\hat{z})\\ p' &= \frac{\sqrt{s}}{2}(1,-\hat{z})\\ k &= \frac{\sqrt{s}}{2}(1,\sin\theta\hat{x}+\cos\theta\hat{z})\\ k' &= \frac{\sqrt{s}}{2}(1,-\sin\theta\hat{x}-\cos\theta\hat{z}).\end{aligned}$$

We can then calculate the Mandelstam variables in terms of $\cos\theta$:

$$\begin{aligned} t &= (p-k)^2 = -2(p\cdot k) = -\frac{s}{2}(1-\cos\theta) = -s\sin^2\frac{\theta}{2} \\ u &= (p-k')^2 = -2(p\cdot k') = -\frac{s}{2}(1+\cos\theta) = -s\cos^2\frac{\theta}{2}\ . \end{aligned}$$

Plugging these into the differential cross section we get

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{\pi\alpha^2}{s}\left[\sin^4\frac{\theta}{2} + \sin^{-4}\frac{\theta}{2} + \cos^4\frac{\theta}{2}\left(1-\sin^{-2}\frac{\theta}{2}\right)^2\right] \\ &= \frac{\pi\alpha^2}{s}\left[\frac{\sin^8(\theta/2)+\cos^8(\theta/2)+1}{\sin^4(\theta/2)}\right] \end{aligned}$$

This is plotted below. The singularity as $\theta \to 0$ is due to the $t$-channel amplitude.

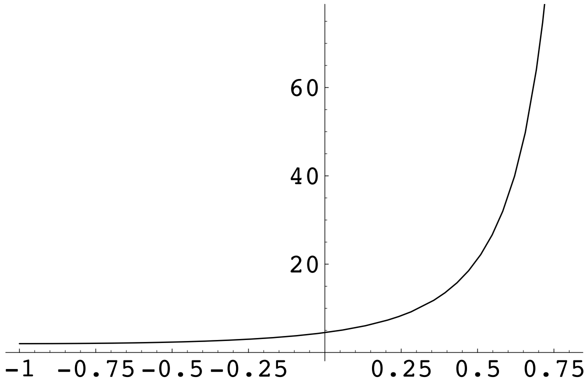


Figure 1: $(s/\pi\alpha^2)d\sigma/d\cos\theta$ vs. $\cos\theta$.

One finds an analogous singularity in the classical calculation due to the infinite range of the Coulomb potential. Screening effects will make this cross section finite at $\theta = 0$ however.