QUANTUM FIELD THEORY I

Physics 443 - Fall Quarter, 2005 - University of Chicago

PROBLEMS DUE TUESDAY, NOVEMBER 15

Problem in text	**Subject**
4–2	Decay Rate [use Eq. (4.86) in text]
4–4	Rutherford scattering

For part (b) in Problem 4–4 it is not necessary to retrace all the steps which led to Eq. (4.79) of the text. Instead, generalize the result to the case in which the target is infinitely heavy and thus is able to absorb 3-momentum but not change energy. If you have not taken a Fourier transform of the Coulomb potential recently, recall that it is most easily done by introducing a cutoff factor $\exp(-\mu r)$ and then taking $\mu \to 0$.

Solutions — Problem Set 6

David McKeen – November 15, 2005

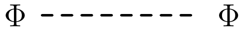
Problem 4–2

We have the Lagrangian

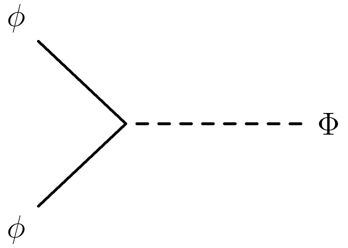
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi.$$

The Feynman rules for the theory are easily written down:

$$= \frac{i}{p^2 - m^2}$$



$$= \frac{i}{p^2 - M^2}$$



$$= 2\mu$$

The factor of two in the vertex comes from contracting the two identical ϕ fields with external states. From this diagram we can immediately read off the amplitude for the process $\Phi \to \phi\phi$:

$$i\mathcal{M} = 2\mu.$$

Eq. (4.86) gives the differential decay rate in terms of the invariant amplitude:

$$d\Gamma = \frac{1}{2}\frac{1}{2m_{\mathcal{A}}}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}|^2\,(2\pi)^4\delta^{(4)}(p_{\mathcal{A}} - \sum p_f).$$

where we have included the factor 1/2 since the final particles are identical. We use Eq. (A.58) to write this in our case as

$$d\Gamma = \frac{1}{4M}\frac{d\Omega}{4\pi}\frac{1}{8\pi}\frac{2\,|\mathbf{p}|}{E_{CM}}\,|\mathcal{M}|^2$$

Using our expression for $\mathcal{M}$ we write this as

$$d\Gamma = \frac{1}{4M}\frac{d\Omega}{4\pi}\frac{1}{8\pi}\frac{2\left|\mathbf{p}\right|}{M}4\mu^2.$$

Integrating we get

$$\Gamma = \frac{\mu^2\left|\mathbf{p}\right|}{4\pi M^2}.$$

Since each ϕ has energy $M/2$ in the center of mass frame we get $\left|\mathbf{p}\right|^2 = M^2/4 - m^2$. Plugging this in we get

$$\Gamma = \frac{\mu^2}{8\pi M}\sqrt{1-4\frac{m^2}{M^2}}$$

or

$$\tau = \frac{8\pi M}{\mu^2\sqrt{1-4\frac{m^2}{M^2}}}.$$

Problem 4–4

(a) The interaction Hamiltonian is

$$H_I = \int d^3x\ e\bar{\psi}\gamma^\mu\psi A_\mu.$$

The T-matrix element is then

$$\begin{aligned}\langle p'|iT|p\rangle &= \langle p'|-i\int dt H_I|p\rangle \\ &= -ie\int d^4x\ \langle p'|\bar{\psi}\gamma^\mu\psi A_\mu|p\rangle.\end{aligned}$$

Our process corresponds to contracting the fermion annihilation operator with the in-state and the fermion creation operator with the out-state, giving a $u(p)e^{-ip\cdot x}$ and a $\bar{u}(p')e^{ip'\cdot x}$ respectively. Therefore

$$\begin{aligned}\langle p'|iT|p\rangle &= -ie\int d^4x\ \bar{u}(p')\gamma^\mu u(p)A_\mu e^{i(p'-p)\cdot x}\\ &= -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p)\end{aligned}$$

with

$$\tilde{A}_\mu(p'-p) = \int d^4x\ A_\mu(x)e^{i(p'-p)\cdot x}.$$

If $A_\mu(x)$ is independent of x^0 then

$$\begin{aligned}\tilde{A}_\mu(p'-p) &= \int dx^0\ e^{i(p_0'-p_0)x^0}\tilde{A}_\mu(\mathbf{p}'-\mathbf{p})\\ &= (2\pi)\delta(E_f-E_i)\tilde{A}_\mu(\mathbf{p}'-\mathbf{p})\end{aligned}$$

with

$$\tilde{A}_\mu(\mathbf{p}'-\mathbf{p}) = \int d^3x\ A_\mu(\mathbf{x})e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}.$$

Then we write

$$\langle p'|iT|p\rangle = i\mathcal{M}(2\pi)\delta(E_f-E_i)$$

with

$$i\mathcal{M} = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(\mathbf{p}'-\mathbf{p}).$$

The discussion of Section 4.5 then holds with this modification of the matrix element.

We can then write

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|\mathcal{M}\right|^2(2\pi)\delta(E_f-E_i).$$

We integrate this over $|\mathbf{p}_f| \equiv p_f$:

$$d\sigma = \frac{d\Omega}{16\pi^2E_iv_i}\int dp_f\frac{p_f^2}{E_f}\left|\mathcal{M}\right|^2\delta(E_f-E_i).$$

since $p_f = \sqrt{E_f^2 - m^2}$, $dp_f = E_f dE_f / \sqrt{E_f^2 - m^2}$. We can then write

$$\begin{aligned} d\sigma &= \frac{d\Omega}{16\pi^2 E_i v_i} \int dE_f \sqrt{E_f^2 - m^2} \left|\mathcal{M}\right|^2 \delta(E_f - E_i) \\ &= \frac{d\Omega}{16\pi^2 E_i v_i} \sqrt{E_i^2 - m^2} \left|\mathcal{M}\right|^2 . \end{aligned}$$

This gives

$$\frac{d\sigma}{d\Omega} = \frac{|\mathbf{p}_i| \left|\mathcal{M}\right|^2}{16\pi^2 E_i v_i}.$$

(c) In the case of a Coulomb potential we have

$$A^0 = \frac{Ze}{4\pi r} \quad A^i = 0.$$

Then,

$$\begin{aligned} \tilde{A}_0(\mathbf{q}) &= \int d^3x \; A_0(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}(1-i\mu)} \\ &= \int d^3x \; \frac{Ze}{4\pi r} e^{-i\mathbf{q}\cdot\mathbf{x}}. \end{aligned}$$

We take the z-axis to coincide with $\mathbf{q}$ and write $\mathbf{q}\cdot\mathbf{x} = q\; r\cos\theta$. We also introduce a slight imaginary component to q which we will take to zero afterwards. Then

$$\begin{aligned} \tilde{A}_0(\mathbf{q}) &= \lim_{\mu\to 0} \frac{Ze}{2} \int_0^\infty dr\; r \int_{-1}^1 d\cos\theta e^{-i(q-i\mu)r\cos\theta} \\ &= \lim_{\mu\to 0} \frac{Ze}{-2i(q-i\mu)} \int_0^\infty dr\; \left(e^{-i(q-i\mu)r} - e^{i(q-i\mu)r}\right) \\ &= \lim_{\mu\to 0} \frac{Ze}{-2i(q-i\mu)} \left(\frac{2}{i(q-i\mu)}\right) \\ &= \frac{Ze}{q^2}. \end{aligned}$$

We know that $|\mathbf{p}_i| = |\mathbf{p}_f| \equiv p$ because of the delta function in energy. Then if the particle is deflected by an angle θ, $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i = p(1-\cos\theta)\hat{z} - p\sin\theta\hat{x}$. This gives

$q^2 = 2p^2\,(1-\cos\theta) = 4p^2\sin^2(\theta/2)$. Then we can write the amplitude as

$$i\mathcal{M} = \frac{-iZe^2\bar{u}(p')\gamma^0 u(p)}{4p^2\sin^2(\theta/2)}.$$

We write

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi \\ \sqrt{p\cdot\bar{\sigma}}\;\xi \end{pmatrix} \to \begin{pmatrix} \sqrt{m}\;\xi \\ \sqrt{m}\;\xi \end{pmatrix}$$

in the nonrelativistic limit. Therefore in the nonrelativistic limit

$$\bar{u}(p')\gamma^0 u(p) \to 2m\xi^\dagger\xi = 2m$$

since ξ is normalized. Using this we get that

$$\begin{aligned} i\mathcal{M} &\to \frac{-iZe^2 m}{2p^2\sin^2(\theta/2)} \\ &= \frac{-2\pi i Z\alpha}{mv^2\sin^2(\theta/2)} \end{aligned}$$

since $p = mv$ in the low energy limit. Looking at the result for the differential cross section we see that in the non relativistic limit it becomes

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2}.$$

Putting this all together we get the desired result

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4\sin^2(\theta/2)^2}.$$