QUANTUM FIELD THEORY I

Physics 443 - Fall Quarter, 2005 - University of Chicago

PROBLEMS DUE TUESDAY, NOVEMBER 1

Problem in text	Subject
3–7	Discrete symmetries C, P, T
3–8	Bound states

In Problem 3.8, S, P, and D refer to relative orbital angular momenta $L = 0,\ 1,\ 2$, respectively. It is convenient to use spectroscopic notation ${}^{2S+1}L_J$, such as $1S_0$, $3P_2$, etc., to refer to the states.

Solutions — Problem Set 4

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Problem 3.7

(a) Using equations (3.145) and (3.146) of Peskin and Schroeder we can write

$$\begin{aligned} C\bar{\psi}\sigma^{\mu\nu}\psi C &= (-i\gamma^0\gamma^2\psi)^T\sigma^{\mu\nu}(-i\bar{\psi}\gamma^0\gamma^2)^T \\ &= -(\gamma^0)_{ab}(\gamma^2)_{bc}\psi_c(\sigma^{\mu\nu})_{ad}\bar{\psi}_f(\gamma^0)_{fe}(\gamma^0)_{ed} \\ &= \bar{\psi}_f(\gamma^0)_{fe}(\gamma^0)_{ed}(\sigma^{\mu\nu})_{ad}(\gamma^0)_{ab}(\gamma^2)_{bc}\psi_c \\ &= \bar{\psi}\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\psi. \end{aligned}$$

We use the fact that $(\gamma^0)^T = \gamma^0$, $(\gamma^1)^T = -\gamma^1$, $(\gamma^2)^T = \gamma^2$, and $(\gamma^3)^T = -\gamma^3$ to write

$$\begin{aligned} (\sigma^{01})^T &= \sigma^{01} & (\sigma^{12})^T &= \sigma^{12} \\ (\sigma^{02})^T &= -\sigma^{02} & (\sigma^{13})^T &= -\sigma^{13} \\ (\sigma^{03})^T &= \sigma^{03} & (\sigma^{23})^T &= \sigma^{23}. \end{aligned}$$

It's then easy to check that

$$\gamma^0\gamma^2(\sigma^{\mu\nu})^T = -\sigma^{\mu\nu}\gamma^0\gamma^2.$$

So,

$$\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2 = -\sigma^{\mu\nu}(\gamma^0\gamma^2)^2$$
$$= -\sigma^{\mu\nu}.$$

Thus we arrive at

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}\sigma^{\mu\nu}\psi.$$

Using equations (3.126) and (3.128) we write

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = \bar{\psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi.$$

One can easily verify that

$$\gamma^0\sigma^{0i} = -\sigma^{0i}\gamma^0$$
$$\gamma^0\sigma^{ij} = \sigma^{ij}\gamma^0$$

or

$$\gamma^0\sigma^{\mu\nu} = (-1)^\mu(-1)^\nu\sigma^{\mu\nu}\gamma^0.$$

This gives us

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = (-1)^\mu(-1)^\nu\bar{\psi}\sigma^{\mu\nu}\psi.$$

Equations (3.123), (3.139), and (3.140) allow us to write

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = -\bar{\psi}\gamma^1\gamma^3\sigma^{*\mu\nu}\gamma^1\gamma^3\psi.$$

One can check that

$$\gamma^1\gamma^3\sigma^{*0i} = \sigma^{0i}\gamma^1\gamma^3.$$

Then

$$T\bar{\psi}\sigma^{0i}\psi T = -\bar{\psi}\sigma^{0i}(\gamma^1\gamma^3)^2\psi$$
$$= \bar{\psi}\sigma^{0i}\psi.$$

It's also easily seen that

$$\gamma^1\gamma^3\sigma^{*ij} = -\sigma^{ij}\gamma^1\gamma^3.$$

Therefore,

$$T\bar{\psi}\sigma^{ij}\psi T = -\bar{\psi}\sigma^{ij}\psi.$$

We summarize this as

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = -(-1)^{\mu}(-1)^{\nu}\bar{\psi}\sigma^{\mu\nu}\psi.$$

(b) As in Problem (2.2) we quantize the complex Klein-Gordon field as

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$
$$\phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

We can realize $P\phi(\mathbf{x},t)P = \phi(-\mathbf{x},t)$ by requiring

$$Pa_{\mathbf{p}}P = a_{-\mathbf{p}} \quad Pb_{\mathbf{p}}P = b_{-\mathbf{p}}.$$

We can also realize $C\phi(x)C = \phi^*(x)$ by requiring

$$Ca_{\mathbf{p}}C = b_{\mathbf{p}}.$$

And, lastly we can realize $T\phi(\mathbf{x},t)T = \phi(\mathbf{x},-t)$ by requiring

$$Ta_{\mathbf{p}}T = a_{-\mathbf{p}} \quad Tb_{\mathbf{p}}T = b_{-\mathbf{p}}$$

and remembering that T is antiunitary, i.e. $T\,(\text{c}-\text{number}) = (\text{c}-\text{number})^*\,T$.

We have the current

$$J^{\mu} = i(\phi^*\partial^{\mu}\phi - \partial^{\mu}\phi^*\phi).$$

Using the transformation rules for ϕ, ϕ^*, and ∂^{μ}, we get that

$$\begin{aligned} CJ^{\mu}C &= i(\phi\partial^{\mu}\phi^* - \partial^{\mu}\phi\phi^*) \\ &= -J^{\mu}. \end{aligned}$$

We also get that

$$\begin{aligned} PJ^{\mu}P &= i(-1)^{\mu}(\phi\partial^{\mu}\phi^* - \partial^{\mu}\phi\phi^*) \\ &= (-1)^{\mu}J^{\mu}. \end{aligned}$$

Lastly,

$$\begin{aligned} TJ^\mu T &= (-i)(-(-1)^\mu)(\phi\partial^\mu\phi^* - \partial^\mu\phi\phi^*) \\ &= (-1)^\mu J^\mu. \end{aligned}$$

(c) The only Lorentz invariant that we can build out of ψ and $\bar{\psi}$ is $\bar{\psi}\psi$. This is seen to have $CPT = +1$ in Peskin and Schroeder. Now, note that if we consider ϕ and ϕ^* in the combination $\phi + \phi^*$ and $i(\phi - \phi^*)$, that is, the real and imaginary components of ϕ, then

$$\begin{aligned} CPT(\phi + \phi^*)TPC &= CP(\phi + \phi^*)PC \\ &= C(\phi^* + \phi)C \\ &= \phi + \phi^* \end{aligned}$$

and

$$\begin{aligned} CPT\left[i(\phi - \phi^*)\right]TPC &= CP\left[-i(\phi - \phi^*)\right]PC \\ &= C\left[-i(\phi - \phi^*)\right]C \\ &= -i\left(\phi^* - \phi\right) \\ &= i\left(\phi - \phi^*\right) \end{aligned}$$

Thus, any operator built from $\bar{\psi}\psi$, Re(ϕ), and Im(ϕ) will have $CPT = +1$.

Problem 3.8

(a) We denote the positronium states as ${}^{2S+1}L_J$ where S is the spin angular momentum of the electron-positron system, L is the orbital angular momentum, and J is the total and angular momentum. In writing down the wavefunction for the hydrogen-like system we can write the spatial component of the wavefunction separately from the spin component. We can therefore consider how the spatial part transforms under C, P, and T separately from the spinor part.

The spatial part of the wavefunction is given by a function of the radius multiplied by a spherical harmonic. It's clear that it's invariant under T since it's a solution of the time independent Schrödinger equation. We only have to consider the effect of a parity flip since the operation of charge conjugation interchanges the position of electron and positron, i.e. it's equivalent to a parity change spatially. The radial portion is only a function of the magnitude of the separation of the electron and positron and is invariant under C and P as well. As for the spherical harmonic, our answer should only depend on L, the total orbital angular momentum and not on m, the magnetic quantum number since different values of m are equivalent up to a rotation of the coordinates. So we need only consider the effect of a parity flip on the Legendre polynomial $P_L(\cos\theta)$. A parity flip is given by $\theta \to \pi - \theta$, $\phi \to \phi + \pi$. Then $\cos\theta \to -\cos\theta$. $P_L(\cos\theta)$ is an even (odd) function of $\cos\theta$ for even (odd) L. Thus a parity inverstion takes $P_L(\cos\theta) \to (-1)^L P_L(\cos\theta)$. By the argument above, charge conjugation also gives a factor of $(-1)^L$.

We now consider the effect of C, P, and T on the spinor component of the wavefunction. So, if $a_{\mathbf{p}}^{s\dagger}$ creates an electron and $b_{\mathbf{p}}^{s\dagger}$ a positron, the effect of charge conjugation is:

$$C a_0^{s\dagger} C = b_0^{s\dagger}$$

we can write the $S = 0$ singlet state at zero momentum as

$$\left(a_0^{1\dagger} b_0^{2\dagger} - a_0^{2\dagger} b_0^{1\dagger}\right) |0\rangle.$$

Under charge conjugation this becomes

$$\left(b_0^{1\dagger} a_0^{2\dagger} - b_0^{2\dagger} a_0^{1\dagger}\right) |0\rangle = \left(a_0^{1\dagger} b_0^{2\dagger} - a_0^{2\dagger} b_0^{1\dagger}\right) |0\rangle$$

where we've used the anticommutation relations between fermionic creation operators. We can write a triplet $S = 1$ state as

$$\left(a_0^{1\dagger} b_0^{2\dagger} + a_0^{2\dagger} b_0^{1\dagger}\right) |0\rangle.$$

Under charge conjugation this becomes

$$\left(b_0^{1\dagger} a_0^{2\dagger} + b_0^{2\dagger} a_0^{1\dagger}\right) |0\rangle = -\left(a_0^{1\dagger} b_0^{2\dagger} + a_0^{2\dagger} b_0^{1\dagger}\right) |0\rangle.$$

It's easy to verify that the other triplet states obey the same relationship. Thus, under C, we pick up a factor $(-1)^S$.

A parity inversion is implemented by

$$P a_{\mathbf{p}}^{s\dagger} P = a_{-\mathbf{p}}^{s\dagger}$$
$$P b_{\mathbf{p}}^{s\dagger} P = -b_{-\mathbf{p}}^{s\dagger}.$$

All electron-positron states therefore pick up a factor of -1 under a parity inversion.

Time reversal is implemented by

$$T a_{\mathbf{p}}^{1\dagger} T = a_{-\mathbf{p}}^{2\dagger}$$
$$T b_{\mathbf{p}}^{1\dagger} T = b_{-\mathbf{p}}^{2\dagger}.$$

The zero momentum singlet state transforms as

$$\left(a_0^{1\dagger} b_0^{2\dagger} - a_0^{2\dagger} b_0^{1\dagger}\right) |0\rangle \rightarrow \left(a_0^{2\dagger} b_0^{1\dagger} - a_0^{1\dagger} b_0^{2\dagger}\right) |0\rangle$$
$$= - \left(a_0^{1\dagger} b_0^{2\dagger} - a_0^{2\dagger} b_0^{1\dagger}\right) |0\rangle.$$

The triplet states are unchanged. Thus we pick up a factor $(-1)^{S+1}$ under T.

Summarizing:

	C	**P**	**T**
${}^1S_0$	1	−1	−1
${}^3S_1$	−1	−1	1
${}^1P_1$	−1	1	−1
${}^3P_{0,1,2}$	1	1	1
${}^1D_2$	1	−1	−1
${}^3D_{1,2,3}$	−1	−1	1
${}^{2S+1}L_J$	$(-1)^{L+S}$	$(-1)^{L+1}$	$(-1)^{S+1}$

(b) The interaction Hamiltonian is given by

$$H_{int} = \int d^3x \; eA_\mu j^\mu = \int d^3x \; eA_\mu \bar{\psi} \gamma^\mu \psi.$$

So, if electrodynamics is to be C, P, and T invariant, A_μ must transform in the same way as $\bar{\psi}\gamma^\mu\psi$ under C, P, and T. In particular in must have odd C-parity. Thus a 2 photon state is even under C and a 3 photon state is odd. Referring to part (a) we see that the ${}^1S_0$ state has even C-parity while ${}^3S_1$ is odd. Therefore the ${}^1S_0$ state can decay to 2 photons but not to 3, while the ${}^3S_1$ state can decay to 3 photons but not to 2.

As for 1-photon transitions between states the fact that photons are odd under C leads to the selection rule $\Delta(L+S) = 2q+1$ with q an integer. The eigenvalues under P and T don't matter because the states can emit electric or magnetic dipole radiation which have opposite parity.