QUANTUM FIELD THEORY I

Physics 443 - Fall Quarter, 2005 - University of Chicago

PROBLEMS DUE TUESDAY, OCTOBER 25

1. Prove the identities (A.27)–(A.29) of the text (p. 805).

2. In many scattering problems you will encounter quantities of the form

$$\mathcal{M} = \mathrm{Tr}(\gamma^5 \not{n} \gamma^\mu \not{p} \gamma^\nu)\mathrm{Tr}(\gamma^5 \not{q} \gamma_\mu \not{k} \gamma_\nu) \quad .$$

Using identities in (A.27) and (A.30), evaluate this matrix element in terms of scalar products of the momenta n, p, q, and k.

3. We will show that the substitition $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$, or equivalently $p_\mu \to p_\mu - eA_\mu$, allows the introduction of electromagnetic interactions. Here e is the electric charge of the particle in question ($e = -|e|$ for an electron), and $A^\mu = (\Phi, \vec{A})$ is the vector potential. By converting the Dirac equation in the form

$$[\vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \beta m]\psi = (E - e\Phi)\psi$$

to a second-order equation, and taking the low-energy limit, show that the interaction with the electromagnetic field gives rise to a change in energy in the presence of a magnetic field $\vec{B} = \nabla \times \vec{A}$ of the form

$$\Delta E = -\frac{e}{2m}\vec{\sigma} \cdot \vec{B}$$

and hence implies a value of $g = 2$ for the electron's magnetic moment $\vec{\mu}$ defined in terms of its spin S as

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S} \ .$$

Solutions — Problem Set 3

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Problem 1

(A.27-1) Since it's the 4×4 unit matrix it's clear that $\mathrm{Tr}(\mathbf{1}) = 4$.

(A.27-2) If we have a trace of n γ matrices:

$$\mathrm{Tr}\left(\gamma^\mu \ldots \gamma^\nu\right) = \mathrm{Tr}\left(\gamma^5\gamma^5\gamma^\mu \ldots \gamma^\rho\right)$$

since $(\gamma^5)^2 = 1$. The cyclic property of the trace then allows us to write

$$\mathrm{Tr}\left(\gamma^\mu \ldots \gamma^\nu\right) = \mathrm{Tr}\left(\gamma^5\gamma^\mu \ldots \gamma^\nu\gamma^5\right).$$

We then commute the γ^5 through the n γ matrices, picking up a factor of -1 with each. We get

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\mu \ldots \gamma^\nu\right) &= (-1)^n \mathrm{Tr}\left(\gamma^5\gamma^5\gamma^\mu \ldots \gamma^\nu\right)\\ &= (-1)^n \mathrm{Tr}\left(\gamma^\mu \ldots \gamma^\nu\right)\end{aligned}$$

Thus, if n is odd, the trace vanishes.

(A.27-3) We can use the Dirac algebra to write

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) &= \mathrm{Tr}\left(2g^{\mu\nu}\mathbf{1} - \gamma^\nu\gamma^\mu\right)\\ &= 8g^{\mu\nu} - \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right).\end{aligned}$$

Therefore

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}.$$

(A.27-4) Using the Dirac algebra, the cyclic property of the trace, and the previous result we can write

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 2g^{\mu\nu}\mathrm{Tr}\left(\gamma^\rho\gamma^\sigma\right) - \mathrm{Tr}\left(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right) \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}\mathrm{Tr}\left(\gamma^\nu\gamma^\sigma\right) + \mathrm{Tr}\left(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\right) \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 2g^{\mu\sigma}\mathrm{Tr}\left(\gamma^\nu\gamma^\rho\right) - \mathrm{Tr}\left(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right) \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\sigma}g^{\nu\rho} - \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right).
\end{aligned}$$

Therefore

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right).$$

(A.27-5) To find $\mathrm{Tr}(\gamma^5)$ we insert a factor of $(\gamma^0)^2 = \mathbf{1}$:

$$\mathrm{Tr}\left(\gamma^5\right) = \mathrm{Tr}\left(\gamma^5\gamma^0\gamma^0\right).$$

We anti-commute the γ^5 past one γ^0:

$$\mathrm{Tr}\left(\gamma^5\right) = -\mathrm{Tr}\left(\gamma^0\gamma^5\gamma^0\right).$$

We then cycle the first γ^0 to the end:

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^5\right) &= -\mathrm{Tr}\left(\gamma^5\gamma^0\gamma^0\right) \\
&= -\mathrm{Tr}\left(\gamma^5\right).
\end{aligned}$$

Therefore

$$\mathrm{Tr}\left(\gamma^5\right) = 0.$$

(A.27-6) To find $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right)$ we make use of the following trick. No matter what μ and ν are there exists a γ^α with $(\gamma^\alpha)^2 = (-1)^\alpha$ and $\alpha \neq \mu, \nu$. We can then insert a factor of

$(\gamma^\alpha)^2$:

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) &= (-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\right) \\
&= -(-1)^\alpha \mathrm{Tr}\left(\gamma^\alpha\gamma^5\gamma^\alpha\gamma^\mu\gamma^\nu\right) \\
&= -(-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\alpha\right) \\
&= -(-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\right) \\
&= -(-1)^\alpha(-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\right) \\
&= -\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\right)
\end{aligned}$$

Therefore

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0.$$

(A.27-7) In considering $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)$, note that if any two indices are the same we can use the same trick as above; i.e. there exists a gamma matrix, γ^α with $(\gamma^\alpha)^2 = (-1)^\alpha$, that will anti-commute with all the gamma matrices in the trace. We then find

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= (-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= -(-1)^\alpha \mathrm{Tr}\left(\gamma^\alpha\gamma^5\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= -(-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\right) \\
&= -(-1)^4(-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= -(-1)^\alpha(-1)^\alpha \mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= -\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right).
\end{aligned}$$

Thus we conclude that

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 0$$

if any two indices are equal. So we can take all 4 indices to be distinct. In that case, all the Dirac matrices anticommute and it's clear that the trace is antisymmetric under interchange of any two indices. That means that we can write

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = C\epsilon^{\mu\nu\rho\sigma}$$

for some constant C. To fix C we evaulate a particular element:

$$\mathrm{Tr}\left(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3\right) = -i\mathrm{Tr}\left(\gamma^5\gamma^5\right) = -4i = -4i\epsilon^{0123}.$$

This gives us

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = -4i\epsilon^{\mu\nu\rho\sigma}.$$

(A.28) To prove this identity we need to demonstrate the following:

$$C\gamma^\mu C = -\left(\gamma^\mu\right)^T, \qquad \text{with } C = \gamma^0\gamma^2.$$

We see this by considering each γ matrix separately:

$$\begin{aligned}
C\gamma^0 C &= \gamma^0\gamma^2\gamma^0\gamma^0\gamma^2 = -\gamma^0 = -\left(\gamma^0\right)^T \\
C\gamma^{1,3} C &= \gamma^0\gamma^2\gamma^{1,3}\gamma^0\gamma^2 \\
&= \gamma^0\gamma^2\gamma^0\gamma^2\gamma^{1,3} = \gamma^{1,3} \\
&= -\left(\gamma^{1,3}\right)^T \\
C\gamma^2 C &= \gamma^0\gamma^2\gamma^2\gamma^0\gamma^2 = -\gamma^2 = -\left(\gamma^2\right)^T.
\end{aligned}$$

It's also clear that $C^2 = 1$. So, if we have a trace of n γ matrices (n even) we can insert factors of C^2 in front of each matrix and then cycle the first C to the end:

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\ldots\right) &= \mathrm{Tr}\left(C^2\gamma^\mu C^2\gamma^\nu\ldots\right) \\
&= \mathrm{Tr}\left[\left(C\gamma^\mu C\right)\left(C\gamma^\nu C\right)\ldots\right] \\
&= (-1)^n\,\mathrm{Tr}\left[\left(\gamma^\mu\right)^T\left(\gamma^\nu\right)^T\ldots\right] \\
&= \mathrm{Tr}\left[\left(\ldots\gamma^\nu\gamma^\mu\right)^T\right] \\
&= \mathrm{Tr}\left(\ldots\gamma^\nu\gamma^\mu\right)
\end{aligned}$$

(A.29-1) It's easy to see that

$$\begin{aligned}\gamma^\mu\gamma_\mu &= (\gamma^0)^2 - (\gamma^1)^2 - (\gamma^2)^2 - (\gamma^3)^2 \\ &= 1 - 3(-1) = 4.\end{aligned}$$

(A.29-2) The anti-commutation relation between γ matrices allows us to write

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= \left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)\gamma_\mu \\ &= 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu.\end{aligned}$$

(A.29-3) We use the anti-commutation relations and the above results to write

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= \left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)\gamma^\rho\gamma_\mu \\ &= 2\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma_\mu \\ &= 2\gamma^\rho\gamma^\nu + 2\gamma^\nu\gamma^\rho \\ &= 4g^{\nu\rho}.\end{aligned}$$

(A.29-4) Similarly,

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= \left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)\gamma^\rho\gamma^\sigma\gamma_\mu \\ &= 2\gamma^\rho\gamma^\sigma\gamma^\nu - 4\gamma^\nu g^{\rho\sigma} \\ &= 4g^{\rho\sigma}\gamma^\nu - 2\gamma^\sigma\gamma^\rho\gamma^\nu - 4\gamma^\nu g^{\rho\sigma} \\ &= -2\gamma^\sigma\gamma^\rho\gamma^\nu.\end{aligned}$$

Problem 2

$$\mathcal{M} = \mathrm{Tr}(\gamma^5\,\not{n}\gamma^\mu\,\not{p}\gamma^\nu)\mathrm{Tr}(\gamma^5\,\not{q}\gamma_\mu\,\not{k}\gamma_\nu)$$

We consider the first trace:

$$\begin{aligned}\mathrm{Tr}(\gamma^5\,\not{n}\gamma^\mu\,\not{p}\gamma^\nu) &= n_\rho p_\sigma \mathrm{Tr}(\gamma^5\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) \\ &= -4i\epsilon^{\rho\mu\sigma\nu} n_\rho p_\sigma\end{aligned}$$

where we used the result from Problem 1. Thus,

$$\mathcal{M} = -16\ \epsilon^{\rho\mu\sigma\nu}\epsilon_{\alpha\mu\beta\nu}\left(n_\rho p_\sigma q^\alpha k^\beta\right).$$

Using the contraction identity from Equation (A.30) of Peskin & Schroeder we see that

$$\begin{aligned}\epsilon^{\rho\mu\sigma\nu}\epsilon_{\alpha\mu\beta\nu} &= \epsilon^{\mu\nu\sigma\rho}\epsilon_{\mu\nu\beta\alpha} \\ &= -2\left(\delta^\sigma_\beta\delta^\rho_\alpha - \delta^\sigma_\alpha\delta^\rho_\beta\right).\end{aligned}$$

Plugging this in we get

$$\mathcal{M} = 32\left[(n\cdot q)(p\cdot k) - (n\cdot k)(p\cdot q)\right].$$

Problem 3

The Dirac equation is

$$\left(i\gamma^\mu D_\mu - m\right)\psi = \left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\psi = 0$$

If we take $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$, $\partial_\mu = (\partial_t, \nabla)$, and $A_\mu = (\Phi, -\mathbf{A})$ then we can write this as

$$\left[i\beta\left(\partial_t + ie\Phi\right) + i\beta\alpha\cdot\left(\nabla - ie\mathbf{A}\right) - m\right]\psi = 0$$

or, since $\beta^2 = 1$,

$$\left[i\left(\partial_t + ie\Phi\right) + i\alpha\cdot\left(\nabla - ie\mathbf{A}\right) - m\beta\right]\psi = 0$$

If we make the usual assignments that $\frac{\partial}{\partial t} \to -iE$ and $\nabla \to i\mathbf{p}$ then we get

$$\left(E - e\Phi\right)\psi = \left(\alpha\cdot\left(\mathbf{p} - e\mathbf{A}\right) + m\beta\right)\psi.$$

Now, pick a particular representation:

$$\beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \alpha_i = \begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}.$$

It's easy to check that these give the correct anti-commutation relations. Then if we denote

$$\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$$

and plug this into the Dirac equation we obtain

$$(E - e\Phi) \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \sigma \cdot (\mathbf{p} - e\mathbf{A}) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + m \begin{pmatrix} \chi \\ -\varphi \end{pmatrix}.$$

If we note that the nonrelativistic energy E' is related to the relativistic by $E' = E - m$ then the equation becomes

$$E' \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \sigma \cdot (\mathbf{p} - e\mathbf{A}) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + e\Phi \begin{pmatrix} \chi \\ \varphi \end{pmatrix} - 2m \begin{pmatrix} 0 \\ \varphi \end{pmatrix}.$$

In the nonrelativistic limit $E' \ll m$ so the second component of the above equation can be written

$$\varphi = \frac{\sigma \cdot (\mathbf{p} - e\mathbf{A})\, \chi}{2m}.$$

We can then write the first component as a second order equation:

$$E'\chi = \left\{ \frac{1}{2m} \sigma \cdot (\mathbf{p} - e\mathbf{A})\, \sigma \cdot (\mathbf{p} - e\mathbf{A}) + e\Phi \right\} \chi.$$

Since $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ we have $(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b})$. So,

$$\begin{aligned}
\sigma \cdot (\mathbf{p} - e\mathbf{A})\, \sigma \cdot (\mathbf{p} - e\mathbf{A}) &= (\mathbf{p} - e\mathbf{A})^2 + i\epsilon^{ijk}\sigma^k \left(-i\partial_i - eA_i\right)\left(-i\partial_j - eA_j\right) \\
&= (\mathbf{p} - e\mathbf{A})^2 + i\epsilon^{ijk}\sigma^k \left(ie\partial_i A_j + ieA_i \partial_j\right) \\
&= (\mathbf{p} - e\mathbf{A})^2 - e\epsilon^{ijk}\sigma^k \left((\partial_i A_j) + A_j \partial_i + A_i \partial_j\right) \\
&= (\mathbf{p} - e\mathbf{A})^2 - e\epsilon^{ijk}\sigma^k \left(\partial_i A_j\right) \\
&= (\mathbf{p} - e\mathbf{A})^2 - e\sigma \cdot (\nabla \times \mathbf{A}) \\
&= (\mathbf{p} - e\mathbf{A})^2 - e\sigma \cdot \mathbf{B}.
\end{aligned}$$

We then get that

$$E'\chi = \left\{ \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - \frac{e\sigma \cdot \mathbf{B}}{2m} + e\Phi \right\} \chi.$$