QUANTUM FIELD THEORY I

Physics 443 - Fall Quarter, 2005 - University of Chicago

PROBLEMS DUE TUESDAY, OCTOBER 18

| **Problem in text** | **Subject** |
|---|---|
| 3–1 | Lorentz group |
| 3–2 | Gordon identity |
| 3–6 | Fierz transformations |

In early editions of the text, there is a misprint in the statement of Problem 3–6 (a). There should be no comma in the equation which normalizes the 16 matrices $\Gamma^A$. See

`http://www.slac.stanford.edu/~mpeskin/QFT.html`

for a complete list of corrections to the text.

## Solutions — Problem Set 2

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# Problem 3.1

**(a)** We use the algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

and define

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i}.$$

We then find that the commutator of rotations is

$$\begin{aligned}
\left[L^i, L^j\right] &= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}\left[J^{kl}, J^{mn}\right] \\
&= -\frac{i}{4}\epsilon^{ikl}\epsilon^{jmn}\left(\delta^{lm}J^{kn} - \delta^{km}J^{ln} - \delta^{ln}J^{km} + \delta^{kn}J^{lm}\right) \\
&= -\frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jln}J^{kn} - \epsilon^{ikl}\epsilon^{jkn}J^{ln} - \epsilon^{ikl}\epsilon^{jml}J^{km} + \epsilon^{ikl}\epsilon^{jmk}J^{lm}\right) \\
&= -iJ^{ji} = iJ^{ij} \\
&= i\epsilon^{ijk}L^k.
\end{aligned}$$

The commutation relation for boosts is

$$\begin{aligned}
\left[K^i, K^j\right] &= \left[J^{0i}, J^{0j}\right] \\
&= i\left(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}\right) \\
&= -iJ^{ij} + i\delta^{ij}J^{00} \\
&= -i\epsilon^{ijk}L^k.
\end{aligned}$$

Commuting a rotation and a boost gives

$$\begin{aligned}
\left[L^i, K^j\right] &= \frac{1}{2}\epsilon^{ikl}\left[J^{kl}, J^{0j}\right] \\
&= \frac{1}{2}\epsilon^{ikl}\left(g^{l0}, J^{kj} - g^{k0}, J^{lj} - g^{lj}, J^{k0} + g^{kj}, J^{l0}\right) \\
&= \frac{1}{2}\left(\epsilon^{ikj}J^{k0} - \epsilon^{ijl}J^{l0}\right) \\
&= \frac{1}{2}\left(\epsilon^{ijk}K^k + \epsilon^{ijl}K^l\right) \\
&= i\epsilon^{ijk}K^k.
\end{aligned}$$

We define $J^i_\pm = L^i \pm iK^i$. The orthogonal combinations commute with each other

$$\begin{aligned}
\left[J^i_+, J^j_-\right] &= \frac{1}{4}\left[L^i + iK^i, L^j - iK^j\right] \\
&= \frac{1}{4}\left\{\left[L^i, L^j\right] - i\left[L^i, K^j\right] + i\left[K^i, L^j\right] + \left[K^i, K^j\right]\right\} \\
&= \frac{1}{4}\left\{i\epsilon^{ijk}L^k - i\left(i\epsilon^{ijk}K^k\right) + i\left(-i\epsilon^{ijk}K^k\right) - i\epsilon^{ijk}L^k\right\} \\
&= 0;
\end{aligned}$$

and individually satisfy the angular momentum commutation relations:

$$\begin{aligned}
\left[J^i_\pm, J^j_\pm\right] &= \frac{1}{4}\left[L^i \pm iK^i, L^j \pm iK^j\right] \\
&= \frac{1}{4}\left\{\left[L^i, L^j\right] \pm i\left[L^i, K^j\right] \pm i\left[K^i, L^j\right] - \left[K^i, K^j\right]\right\} \\
&= \frac{1}{4}\left\{i\epsilon^{ijk}L^k \pm i\left(i\epsilon^{ijk}K^k\right) \pm i\left(i\epsilon^{ijk}K^k\right) + i\epsilon^{ijk}L^k\right\} \\
&= \frac{i}{2}\epsilon^{ijk}\left(L^k \pm iK^k\right) \\
&= i\epsilon^{ijk}J^k_\pm.
\end{aligned}$$

**(b)** For the $\left(\frac{1}{2}, 0\right)$ representation we want $J^i_+ = \frac{\sigma^i}{2}$ and $J^i_+ = 0$. We can accomplish this by taking $L^i = iK^i$. This gives us $L^i = \frac{\sigma^i}{2}$ and $K^i = -\frac{i\sigma^i}{2}$. We plug into the transformation law for $\Phi$, getting

$$\begin{aligned}
\Phi_{\left(\frac{1}{2},0\right)} &\to \left(1 - i\theta^i L^i - i\beta^i K^i\right)\Phi \\
&= \left(1 - i\theta^i\frac{\sigma^i}{2} - \beta^i\frac{\sigma^i}{2}\right)\Phi
\end{aligned}$$

transforming as does $\psi_L$ in Eq. (3.37) of Peskin and Schroeder.

For the $\left(0,\frac{1}{2}\right)$ representation we want $J_+^i = 0$ and $J_+^i = \frac{\sigma^i}{2}$. We take $L^i = -iK^i = \frac{\sigma^i}{2}$. We plug into the transformation law for $\Phi$, getting

$$\Phi_{\left(0,\frac{1}{2}\right)} \to \left(1 - i\theta^i\frac{\sigma^i}{2} + \beta^i\frac{\sigma^i}{2}\right)\Phi$$

transforming as does $\psi_R$ in Eq. (3.37).

**(c)** We write

$$\Phi_{\left(\frac{1}{2},\frac{1}{2}\right)} = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu\bar{\sigma}_\mu.$$

We then see that

$$\begin{aligned}
\Phi_{\left(\frac{1}{2},\frac{1}{2}\right)} &\to \left(1 - i\theta^i\frac{\sigma^i}{2} + \beta^i\frac{\sigma^i}{2}\right)V^\mu\bar{\sigma}_\mu\left(1 + i\theta^i\frac{\sigma^i}{2} + \beta^i\frac{\sigma^i}{2}\right) \\
&= V^\mu\bar{\sigma}_\mu - i\theta^i\frac{\sigma^i}{2}V^\mu\bar{\sigma}_\mu + \beta^i\frac{\sigma^i}{2}V^\mu\bar{\sigma}_\mu + iV^\mu\bar{\sigma}_\mu\theta^i\frac{\sigma^i}{2} + V^\mu\bar{\sigma}_\mu\beta^i\frac{\sigma^i}{2} + \ldots \\
&= V^\mu\bar{\sigma}_\mu - \frac{i}{2}V^\mu\theta^i\left[\sigma^i,\bar{\sigma}_\mu\right] + \frac{1}{2}V^\mu\beta^i\left\{\sigma^i,\bar{\sigma}_\mu\right\} + \ldots \\
&= V^\mu\bar{\sigma}_\mu - \frac{i}{2}V^0\theta^i\left[\sigma^i,\sigma^0\right] + \frac{i}{2}V^j\theta^i\left[\sigma^i,\sigma^j\right] + \frac{1}{2}V^0\beta^i\left\{\sigma^i,\sigma^0\right\} - \frac{1}{2}V^j\beta^i\left\{\sigma^i,\sigma^j\right\} + \ldots
\end{aligned}$$

We have

$$\left[\sigma^i,\sigma^0\right] = 0 \qquad\qquad \left[\sigma^i,\sigma^j\right] = 2i\epsilon^{ijk}\sigma^k$$
$$\left\{\sigma^i,\sigma^0\right\} = 2\sigma^i \qquad\qquad \left\{\sigma^i,\sigma^j\right\} = -2\delta^{ij}$$

Using this we get

$$\Phi_{\left(\frac{1}{2},\frac{1}{2}\right)} \to V^\mu\bar{\sigma}_\mu + \frac{i}{2}V^j\theta^i\left(2i\epsilon^{ijk}\sigma^k\right) + \frac{1}{2}V^0\beta^i\left(2\sigma^i\right) - \frac{1}{2}V^j\beta^i\left(-2\delta^{ij}\right)$$

$$= V_\mu\bar{\sigma}^\mu - \epsilon^{ijk}\theta^iV^j\sigma^k + \beta^iV^0\sigma^i + \beta^iV^i. \tag{1}$$

Now, we wish to show that this transformation law for $V_\mu\sigma^\mu = \Phi_{\left(\frac{1}{2},\frac{1}{2}\right)}$ is the same as what we'd get by starting with the assumption that $V_\mu$ transforms as a 4-vector. In that case,

$$V_\mu\bar{\sigma}^\mu \to V_\mu\bar{\sigma}^\mu - \frac{i}{2}\omega_{\alpha\beta}\left(J^{\alpha\beta}\right)_{\mu\nu}V^\nu\bar{\sigma}^\mu$$

with

$$\left(J^{\alpha\beta}\right)_{\mu\nu} = i\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right).$$

Using this for for $\left(J^{\alpha\beta}\right)_{\mu\nu}$ we get

$$\begin{aligned}
V_\mu\bar{\sigma}^\mu &\to V_\mu\bar{\sigma}^\mu + \frac{1}{2}\omega_{\alpha\beta}\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right)V^\nu\bar{\sigma}^\mu \\
&= V_\mu\bar{\sigma}^\mu + \omega_{\mu\nu}V^\nu\bar{\sigma}^\mu \\
&= V_\mu\bar{\sigma}^\mu + \omega_{0i}V^i - \omega_{i0}V^0\sigma^i - \omega_{ij}V^j\sigma^i \\
&= V_\mu\sigma^\mu + \beta^i V^i + \beta^i V^0\sigma^i - \epsilon^{ijk}\theta^k V^j\sigma^i.
\end{aligned}$$

where we have used $\omega_{0i} = -\omega_{i0} = \beta^i$ and $\omega_{ij} = \omega^{ij} = \epsilon^{ijk}\theta^k$. We see that this is the same transformation as in Equation (1). We conclude that $V^\mu$ transforms as a 4-vector.

# Problem 3.2

Using the definitions of $\sigma^{\mu\nu}$ and $q^\mu$ we see that

$$\frac{i\sigma^{\mu\nu}}{2m}q_\nu = -\frac{1}{4m}\left(\gamma^\mu\gamma^\nu p'_\nu - \gamma^\mu\gamma^\nu p'_\nu - \gamma^\nu\gamma^\mu p_\nu + \gamma^\nu\gamma^\mu p_\nu\right).$$

Using $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$ we write this as

$$\begin{aligned}
\frac{i\sigma^{\mu\nu}}{2m}q_\nu &= -\frac{1}{2m}\left(g^{\mu\nu}p'_\nu - \gamma^\nu\gamma^\mu p'_\nu - \gamma^\mu\gamma^\nu p_\nu + g^{\nu\mu}p_\nu\right) \\
&= \frac{1}{2m}\left(\not{p}'\gamma^\mu + \gamma^\mu\not{p} - p'^\mu - p^\mu\right).
\end{aligned}$$

The Dirac equation gives $\not{p}u(p) = mu(p) \Rightarrow \bar{u}(p)\not{p} = m\bar{u}(p)$. Using these results we can write

$$\begin{aligned}
\bar{u}(p')\frac{i\sigma^{\mu\nu}}{2m}q_\nu u(p) &= \frac{1}{2m}\bar{u}(p')\left(m\gamma^\mu + \gamma^\mu m - p'^\mu - p^\mu\right)u(p) \\
&= \bar{u}(p')\gamma^\mu u(p) - \frac{1}{2m}\bar{u}(p')\left(p'^\mu + p^\mu\right)u(p).
\end{aligned}$$

Rearranging this we get the Gordon identity:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p^\mu + p'^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p).$$

# Problem 3.6

**(a)** One can verify that the following 16 $4\times 4$ matrices satisfy $\mathrm{tr}\left[\Gamma^A\Gamma^B\right] = 4\delta^{AB}$:

$$\begin{array}{llll}
\Gamma^1 = 1 & \Gamma^5 = i\gamma^3 & \Gamma^9 = \sigma^{12} & \Gamma^{13} = \gamma^1\gamma^5 \\
\Gamma^2 = \gamma^0 & \Gamma^6 = i\sigma^{01} & \Gamma^{10} = \sigma^{13} & \Gamma^{14} = \gamma^2\gamma^5 \\
\Gamma^3 = i\gamma^1 & \Gamma^7 = i\sigma^{02} & \Gamma^{11} = \sigma^{23} & \Gamma^{15} = \gamma^3\gamma^5 \\
\Gamma^4 = i\gamma^2 & \Gamma^8 = i\sigma^{03} & \Gamma^{12} = i\gamma^0\gamma^5 & \Gamma^{16} = \gamma^5.
\end{array}$$

**(b)** Note: All repeated indices labeling matrix elements are assumed to be summed over. The matrices are orthogonal and complete,

$$\mathrm{tr}\left[\Gamma^A\Gamma^B\right] = \Gamma^A_{ij}\Gamma^B_{ji} = 4\delta^{AB}$$
$$W_{ij} = \sum_A c^A\Gamma^A_{ij}.$$

where $W$ is an arbitrary $4\times 4$ matrix. We can use these relations to write

$$c^A = \frac{1}{4}\mathrm{tr}\left(W\Gamma^A\right)$$

or

$$W_{ij} = \frac{1}{4}\sum_A \Gamma^A_{ij}\Gamma^A_{kl}W_{lk},$$

From this we can see that

$$\frac{1}{4}\sum_A \Gamma^A_{ij}\Gamma^A_{kl} = \delta_{il}\delta_{jk}.$$

Now,

$$\begin{aligned}
\left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right) &= \bar{u}^i_1 u^j_2 \bar{u}^k_3 u^l_4 \Gamma^A_{ij}\Gamma^B_{kl} \\
&= \left(\delta_{i\alpha}\delta_{l\delta}\right)\left(\delta_{j\beta}\delta_{k\gamma}\right)\bar{u}^i_1 u^j_2 \bar{u}^k_3 u^l_4 \Gamma^A_{\alpha\beta}\Gamma^B_{\gamma\delta} \\
&= \frac{1}{16}\sum_{C,D}\Gamma^C_{il}\Gamma^C_{\delta\alpha}\Gamma^D_{kj}\Gamma^D_{\beta\gamma}\bar{u}^i_1 u^j_2 \bar{u}^k_3 u^l_4 \Gamma^A_{\alpha\beta}\Gamma^B_{\gamma\delta} \\
&= \frac{1}{16}\sum_{C,D}\left(\Gamma^C_{\delta\alpha}\Gamma^A_{\alpha\beta}\Gamma^D_{\beta\gamma}\Gamma^B_{\gamma\delta}\right)\bar{u}^i_1 u^j_2 \bar{u}^k_3 u^l_4 \Gamma^C_{il}\Gamma^D_{kj} \\
&= \frac{1}{16}\sum_{C,D}\mathrm{tr}\left(\Gamma^C\Gamma^A\Gamma^D\Gamma^B\right)\bar{u}^i_1 u^j_2 \bar{u}^k_3 u^l_4 \Gamma^C_{il}\Gamma^D_{kj} \\
&= \sum_{C,D} C^{AB}{}_{CD}\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^D u_2\right)
\end{aligned}$$

with

$$C^{AB}{}_{CD} = \frac{1}{16}\mathrm{tr}\left(\Gamma^C\Gamma^A\Gamma^D\Gamma^B\right).$$

**(c)** In the case of $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ we have $\Gamma^A = \Gamma^B = 1$ and so we have

$$C^{AB}{}_{CD} = \frac{1}{16}\mathrm{tr}\left(\Gamma^C\Gamma^D\right) = \frac{1}{4}\delta^{CD}$$

Thus we have

$$
\begin{aligned}
(\bar{u}_1 u_2)\,(\bar{u}_3 u_4) &= \frac{1}{4}\sum_C \left(\bar{u}_1 \Gamma^C u_4\right)\left(\bar{u}_3 \Gamma^C u_2\right) \\
&= \frac{1}{4}\Big\{(\bar{u}_1 u_4)\,(\bar{u}_3 u_2) + \left(\bar{u}_1 \gamma^0 u_4\right)\left(\bar{u}_3 \gamma^0 u_2\right) - \left(\bar{u}_1 \gamma^i u_4\right)\left(\bar{u}_3 \gamma^i u_2\right) \\
&\qquad - \left(\bar{u}_1 \sigma^{0i} u_4\right)\left(\bar{u}_3 \sigma^{0i} u_2\right) + \left(\bar{u}_1 \sigma^{ij} u_4\right)\left(\bar{u}_3 \sigma^{ij} u_2\right)_{i<j} - \left(\bar{u}_1 \gamma^0\gamma^5 u_4\right)\left(\bar{u}_3 \gamma^0\gamma^5 u_2\right) \\
&\qquad + \left(\bar{u}_1 \gamma^0\gamma^i u_4\right)\left(\bar{u}_3 \gamma^0\gamma^i u_2\right) + \left(\bar{u}_1 \gamma^5 u_4\right)\left(\bar{u}_3 \gamma^5 u_2\right)\Big\} \\
&= \frac{1}{4}\,(\bar{u}_1 u_4)\,(\bar{u}_3 u_2) + \frac{1}{4}\,(\bar{u}_1 \gamma^\mu u_4)\,(\bar{u}_3 \gamma_\mu u_2) + \frac{1}{8}\,(\bar{u}_1 \sigma^{\mu\nu} u_4)\,(\bar{u}_3 \sigma_{\mu\nu} u_2) \\
&\qquad - \frac{1}{4}\left(\bar{u}_1 \gamma^\mu\gamma^5 u_4\right)\left(\bar{u}_3 \gamma_\mu\gamma^5 u_2\right) + \frac{1}{4}\left(\bar{u}_1 \gamma^5 u_4\right)\left(\bar{u}_3 \gamma^5 u_2\right)
\end{aligned}
$$

To get the Fierz identity for $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$ we note that it transforms as a Lorentz scalar. This means that each term in the decomposition must also transform as a Lorentz scalar. Thus we can write:

$$
\begin{aligned}
(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4) = {}& \alpha(\bar{u}_1 u_4)(\bar{u}_3 u_2) + \beta(\bar{u}_1\gamma^\mu u_4)(\bar{u}_3\gamma_\mu u_2) + \gamma(\bar{u}_1\sigma^{\mu\nu} u_4)(\bar{u}_3\sigma_{\mu\nu} u_2) \\
& + \delta(\bar{u}_1\gamma^\mu\gamma^5 u_4)(\bar{u}_3\gamma_\mu\gamma^5 u_2) + \rho(\bar{u}_1\gamma^5 u_4)(\bar{u}_3\gamma^5 u_2).
\end{aligned}
$$

First note that

$$
\begin{aligned}
\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu &= \left(2g^{\mu\rho} - \gamma^\rho\gamma^\mu\right)\left(2\delta^\sigma_\mu - \gamma_\mu\gamma^\sigma\right) \\
&= 4g^{\mu\rho}\delta^\sigma_\mu - 2g^{\mu\rho}\gamma_\mu\gamma^\sigma - 2\delta^\sigma_\mu\gamma^\rho\gamma^\mu + \gamma^\rho\gamma^\mu\gamma_\mu\gamma^\sigma \\
&= 4g^{\sigma\rho}.
\end{aligned}
$$

So, the coefficients of the $\sigma^{\mu\nu}$ terms involve $\mathrm{tr}\left(\sigma^{\rho\sigma}\gamma^\mu\sigma_{\rho\sigma}\gamma_\mu\right) \propto g_{\rho\sigma} = 0$ since $\rho \neq \sigma$. Thus we set $\gamma = 0$ in the decomposition above. The remaining coefficients can be calculated in terms of the traces involving the time component only. We get

$$
\begin{aligned}
\alpha &= \frac{1}{16}\mathrm{tr}\left(\gamma^\mu\gamma_\mu\right) = \frac{1}{4}\mathrm{tr}\left(1\right) = 1. \\
\beta &= \frac{1}{16}\mathrm{tr}\left(\gamma^0\gamma^\mu\gamma^0\gamma_\mu\right) \\
&= -\frac{1}{8}\mathrm{tr}\left(\gamma^0\gamma^0\right) = -\frac{1}{2} \\
\delta &= \frac{1}{16}\mathrm{tr}\left[\left(\gamma^0\gamma^5\right)\gamma^\mu\left(\gamma^0\gamma^5\right)\gamma_\mu\right] \\
&= \frac{1}{16}\mathrm{tr}\left(\gamma^0\gamma^\mu\gamma^0\gamma_\mu\right) = -\frac{1}{2} \\
\rho &= \frac{1}{16}\mathrm{tr}\left(\gamma^5\gamma^\mu\gamma^5\gamma_\mu\right) \\
&= -\frac{1}{16}\mathrm{tr}\left(\gamma^\mu\gamma_\mu\right) = -1.
\end{aligned}
$$

Thus we can write

$$
(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4) = (\bar{u}_1 u_4)(\bar{u}_3 u_2) - \frac{1}{2}(\bar{u}_1\gamma^\mu u_4)(\bar{u}_3\gamma_\mu u_2) - \frac{1}{2}(\bar{u}_1\gamma^\mu\gamma^5 u_4)(\bar{u}_3\gamma_\mu\gamma^5 u_2) - (\bar{u}_1\gamma^5 u_4)(\bar{u}_3\gamma^5 u_2).
$$