ELEMENTARY PARTICLE PHYSICS

Physics 363 - Spring Quarter, 2007 - University of Chicago

PROBLEM DUE THURSDAY, APRIL 26

In our notation, the Dirac field has the expansion (three-vectors are denoted $\vec{p}$, etc.):

$$\psi_\alpha(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{\vec{p},s} \frac{1}{\sqrt{2E_p}} [b_{\vec{p},s} u_\alpha(\vec{p},s) e^{-ip\cdot x} + d^\dagger_{\vec{p},s} v_\alpha(\vec{p},s) e^{ip\cdot x}] \ , \qquad (1)$$

where the Dirac spinors are

$$u(\vec{p},s) = \sqrt{E_p+m} \begin{bmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s \end{bmatrix} \ , \quad v(\vec{p},s) = \sqrt{E_p+m} \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi_s \\ \xi_s \end{bmatrix} \ , \quad (2)$$

and χ_s, ξ_s are two-component Pauli spinors for particles and antiparticles. We showed in class that the Hamiltonian could be written

$$H = \sum_{\vec{p},s} E_p (b^\dagger_{\vec{p},s} b_{\vec{p},s} + d^\dagger_{\vec{p},s} d_{\vec{p},s}) \ . \qquad (3)$$

(1) Find the expansion in terms of momentum-space creation and annihilation operators for the three-momentum operator

$$\vec{P} = \int d^3x \ \psi^\dagger (-i\vec{\nabla}) \psi \ . \qquad (4)$$

(2) Find the corresponding mode expansion for the charge operator

$$Q = \int d^3x \ \psi^\dagger \psi \ . \qquad (5)$$

ANSWERS

(1) In the expression (4) for the momentum operator, there will be a double mode sum. The field operator ψ contributes a plane wave factor $e^{i\vec{p}\cdot\vec{x}}$, on which $-i\vec{\nabla}$ gives a factor $\vec{p}$, multiplying the product $b_{\vec{p},s} u_\alpha(\vec{p},s)$, and a plane wave factor $e^{-i\vec{p}\cdot\vec{x}}$, on which $-i\vec{\nabla}$ gives a factor $-\vec{p}$, multiplying the product $d^\dagger_{\vec{p},s} v_\alpha(\vec{p},s)$. As in the calculation of the Hamiltonian H, we use the identities

$$\int d^3x e^{\pm i(\vec{p}\,'-\vec{p})\cdot\vec{x}} = V \delta_{\vec{p}\,',\vec{p}} \ , \quad \int d^3x e^{\pm i(\vec{p}\,'+\vec{p})\cdot\vec{x}} = V \delta_{\vec{p}\,',-\vec{p}} \ . \qquad (6)$$

Then the double mode sum over $\vec{p}$ and $\vec{p}'$ reduces to a single mode sum. Moreover, as in calculating H, we may use the identities

$$u^\dagger(\vec{p},s')u(\vec{p},s) = v^\dagger(\vec{p},s')v(\vec{p},s) = 2E\delta_{ss'} \ , \quad v^\dagger(-\vec{p},s')u(\vec{p},s) = u^\dagger(-\vec{p},s')v(\vec{p},s) = 0 \ . \tag{7}$$

Thus we are left with

$$\vec{P} = \sum_{\vec{p},s} \vec{p}[b^\dagger_{\vec{p},s}b_{\vec{p},s} - d_{\vec{p},s}d^\dagger_{\vec{p},s}] = \sum_{\vec{p},s} \vec{p}[b^\dagger_{\vec{p},s}b_{\vec{p},s} + d^\dagger_{\vec{p},s}d_{\vec{p},s}] \ , \tag{8}$$

where the effect of the constant gotten by interchanging the d and $d^\dagger$ operators vanishes when summed over all $\vec{p}$.

(2) Here one uses the same results as in part (1) but the calculation is simpler as there is no $-i\vec{\nabla}$ operator. The result is then

$$Q = \sum_{\vec{p},s}[b^\dagger_{\vec{p},s}b_{\vec{p},s} + d_{\vec{p},s}d^\dagger_{\vec{p},s}] \tag{9}$$

which can be rewritten, subtracting an infinite constant which may be thought of as the charge of the filled Dirac sea, as

$$Q = \sum_{\vec{p},s}[b^\dagger_{\vec{p},s}b_{\vec{p},s} - d^\dagger_{\vec{p},s}d_{\vec{p},s}] \ . \tag{10}$$

This last result also can be obtained immediately if one is careful to define the charge in terms of normal-ordered operators as

$$Q = \int d^3x \ : \ \psi^\dagger\psi \ : \ . \tag{11}$$