ELEMENTARY PARTICLE PHYSICS

Physics 363 - Spring Quarter, 2007 - University of Chicago

PROBLEMS DUE THURSDAY, APRIL 19

(1) The *rapidity* of a particle with momentum $p^{||}$ parallel to some axis and energy p^0 is defined by

$$y \equiv \frac{1}{2} \ln \frac{p^0 + p^{||}}{p^0 - p^{||}} \quad . \tag{1}$$

Show that under a boost characterized by a velocity β and a Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$, the rapidity changes by an additive constant χ, where $\gamma = \cosh \chi$.

(2) For a 2-body decay $A \to B + C$, calculate the magnitude of the final center-of-mass 3-momentum p^* of either particle.

(3) For the decay of a neutral pion with laboratory energy E_{π^0} into two photons, $\pi^0 \to \gamma\gamma$, show that there is a minimum opening angle $\theta_{12}^{\rm min}$ between the photons and calculate its value in terms of E_{π^0} and m_{π^0}.

(4) The neutral pion is spinless, so the distribution of photons in $\pi^0 \to \gamma\gamma$ is isotropic in the pion's rest frame. Let the pion now be considered in a frame with Lorentz boost velocity β and Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$. Show that the distribution of photon energies is uniform in the lab frame, and calculate its lower and upper limits in terms of m_{π^0}, β, and γ.

ANSWERS

(1) Use the fact that the "boosted" energy p'^0 and parallel component of momentum $p'^{||}$ are related to p^0 and $p^{||}$ by $p'^0 = \gamma(p^0 + \beta p^{||})$, $p'^{||} = \gamma(p^{||} + \beta p^0)$. Then

$$y' \equiv \frac{1}{2} \ln \frac{\gamma(p^0 + \beta p^{||}) + \gamma(p^{||} + \beta p^0)}{\gamma(p^0 + \beta p^{||}) - \gamma(p^{||} + \beta p^0)} = \frac{1}{2} \ln \frac{(p^0 + p^{||})(1+\beta)}{(p^0 - p^{||})(1-\beta)} = y + \frac{1}{2} \ln \frac{1+\beta}{1-\beta} \quad . \tag{2}$$

Optional: this can be expressed in terms of an additive shift $y' = y + \chi$ where

$$\chi \equiv \frac{1}{2}\ln\frac{1+\beta}{1-\beta}~, \quad e^{\chi} = \sqrt{\frac{1+\beta}{1-\beta}}~. \tag{3}$$

Suppose we define a hyperbolic angle ϕ such that $\cosh\phi = \gamma = (1-\beta^2)^{-1/2}$. Then $\sinh\phi = (1-\cosh^2\phi)^{1/2} = \beta/(1-\beta^2)^{1/2}$, $e^{\phi} = \cosh\phi + \sinh\phi = (1+\beta)/(1-\beta^2)^{1/2} = \sqrt{(1+\beta)/(1-\beta)}$, so $\chi = \phi$ and $y' = y + \cosh^{-1}\gamma$.

(2) Take invariant squares of 4-momenta: $M_A^2 = p_A^2 = (p_B + p_C)^2 = M_B^2 + 2p_B\cdot p_C + M_B^2$. Since the three-momenta of p_B and p_C are equal and opposite in the center-of-mass of A, their magnitudes p^* enter into the scalar product as

$$2p_B\cdot p_C = 2E_BE_C + 2p^{*2} = 2\sqrt{p^{*2}+M_B^2}\sqrt{p^{*2}+M_C^2} + 2p^{*2} = M_A^2 - M_B^2 - M_C^2~. \tag{4}$$

Now move $2p^{*2}$ to the right-hand side and square both sides to get rid of the square roots. After cancelling equal terms on both sides, one finds

$$p^* = \frac{1}{2M_A}\lambda^{1/2}(M_A^2, M_B^2, M_C^2)~, \quad \lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz~. \tag{5}$$

(3) We have $m_{\pi^0}^2 = (p_1 + p_2)^2 = 2E_1E_2(1-\cos\theta_{12}) = 4E_1E_2\sin^2(\theta_{12}/2)$, where $p_{1,2}$ and $E_{1,2}$ are the four-momenta and laboratory energies of the two photons. Now $E_1 + E_2 = E_{\pi^0}$ is fixed, so the minimum angle θ_{12} is attained for the maximum value of E_1E_2 subject to fixed $E_1 + E_2$, which occurs when $E_1 = E_2 = E_{\pi^0}/2$. Thus $m_{\pi^0}^2 = E_{\pi^0}^2\sin^2\theta_{12}^{\rm min}/2$, or $\sin\theta_{12}^{\rm min}/2 = m_{\pi^0}/E_{\pi^0}$.

(4) In the π^0 center of mass, each photon has an energy $E^* = m_{\pi^0}/2$. Let θ^* be the angle of photon 1 with respect to the boost direction; the angle of photon 2 will be $\pi - \theta^*$. The momentum components of the two photons in this frame parallel to the boost axis are $\pm(m_{\pi^0}/2)\cos\theta^*$. When boosted by a velocity β, the photon energies are $E_1 = \gamma(E^*)(1+\beta\cos\theta^*)$, $E_2 = \gamma(E^*)(1-\beta\cos\theta^*)$. Since the distribution in $\cos\theta^*$ is uniform, each photon is uniformly distributed in laboratory energy E from $\gamma(m_{\pi^0}/2)(1-\beta)$ to $\gamma(m_{\pi^0}/2)(1+\beta)$.