ELEMENTARY PARTICLE PHYSICS

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PROBLEM DUE THURSDAY, APRIL 12

The spacing between the $\Upsilon(2S)$ and $\Upsilon(1S)$ $^3S_1$ $b\bar{b}$ (bottomonium) levels is about 560 MeV, while that between the $\psi'(2S)$ and $J/\psi(1S)$ $^3S_1$ $c\bar{c}$ (charmonium) levels is about 589 MeV. For what form of interquark potential $V(r)$ in the radial Schrödinger equation

$$\left[-\frac{1}{m_Q}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{m_Q r^2} + V(r)\right]u(r) = Eu(r) \tag{1}$$

would the level spacing be *independent* of m_Q? (Show this, for example, by redefining a scaled variable $\rho \equiv r\sqrt{m_Q}$ and writing the radial Schrödinger equation in terms of ρ. Here $u(r)$ is the reduced radial wave function, behaving as $r^{\ell+1}$ as $r \to 0$.)

ANSWER

Writing the equation in terms of $\rho \equiv r\sqrt{m_Q}$, the only m_Q dependence then appears in $V(r) = V(\rho/\sqrt{m_Q})$. Under a change in m_Q by a multiplicative factor, $m_Q \to \lambda m_Q$, we want $V(r)$ to shift by a constant:

$$V(\rho/\sqrt{\lambda m_Q}) = V(\rho/\sqrt{m_Q}) + \Delta(\lambda) \quad . \tag{2}$$

Partial-differentiate both sides with respect to λ:

$$-\frac{\rho}{2\lambda^{3/2}\sqrt{m_Q}}V'(\rho/\sqrt{\lambda m_Q}) = \frac{\partial\Delta}{\partial\lambda} \tag{3}$$

and note that the right-hand side is *independent of* ρ. Then the left-hand side must also be independent of ρ, which can only be satisfied if $V'(\rho/\sqrt{\lambda m_Q}) = C/\rho$, where C is some constant. Then one must have $V(r) = C\ln(r/r_0)$, where r_0 is another constant.

This can also be solved by assuming a power-law potential $V(r) \sim r^\alpha$ and determining the dependence of level splittings on m_Q as a function of α. This again relies on absorbing a suitable power of m_Q into the definition of the independent variable so that the Schrödinger equation takes on a universal form. The result is that energy level spacings scale as $m_Q^{-\alpha/(2+\alpha)}$. They are indepdendent of m_Q for $\alpha = 0$, which would be a constant potential. However, one can also regard $V(r) = C\ln(r/r_0)$ as the limit of a power-law potential as $\alpha \to 0$, with a coupling constant growing as $1/\alpha$ and a constant also growing as $1/\alpha$ subtracted off. See C. Quigg and J. L. Rosner, Phys. Lett. 71B, 153 (1977).