Final project for Physics 342 - Quantum Mechanics II

BMT Equation Analysis and Spin in Accelerator Physics

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Abstract.

As known from the non relativistic quantum mechanics, the spin motion in any magnetic field is a precession due to an interaction of a magnetic moment associated with spin with this field. A generalization of this approach to the relativistic case leads to so called BMT-equation, which describes the spin motion in arbitrary electric and magnetic fields.

Some aspects of the dynamics of a spin-polarized beams in particle accelerators are discussed in this paper. Some fundament of its theory is derived starting from the basic principles of quantum mechanics as well as classical mechanicsare and accelerator physics. The Bargmann-Michel-Telegdi equation derived using Thomas approach. The breif analysis of the BMT equation is also presented.

1. Classical Spin Model

The equation of motion for the expectation value is given by one of the most fundamental theorem of quantum mechanics — the Ehrenfest's theorem (Ehrenfest 1927). In most general case, it states that for a system in a state $|\psi\rangle$ the expectation value of a quantum operator $\hat{A}$ satisfies to:

$$\frac{\mathrm{d}\langle\hat{A}\rangle}{\mathrm{d}t} = \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{1}{\mathrm{i}\hbar}\langle[\hat{A},\hat{H}]\rangle, \tag{1.1}$$

where $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$ and $\hat{H}$ is the Hamiltonian operator.

Consider the application of this theorem for the case when $\hat{A}$ stands for the spin operator $\hat{\boldsymbol{s}}$ and the Hamiltonian is in the form (magnetic dipole Hamiltonian)

$$\hat{H} = -\hat{\boldsymbol{\mu}}\cdot\hat{\boldsymbol{B}} = \hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{s}}, \tag{1.2}$$

where $\hat{\boldsymbol{B}}$ is the magnetic field and $\hat{\boldsymbol{\Omega}}$ is the operator of spin precession vector. Using the commutation relation $[\hat{\boldsymbol{s}},\hat{H}] = \mathrm{i}\hbar\,\hat{\boldsymbol{\Omega}}\times\hat{\boldsymbol{s}}$ and the fact that $\partial\hat{A}/\partial t = 0$, one can derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{s}}\rangle = \langle\hat{\boldsymbol{\Omega}}\times\hat{\boldsymbol{s}}\rangle. \tag{1.3}$$

When the spin precession vector can be represented as $\hat{\boldsymbol{\Omega}} = \Omega\,\boldsymbol{e}$, where $\boldsymbol{e}$ is a fixed unit vector, and it is independent of time, one can rewrite (1.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{s}}\rangle = \Omega\boldsymbol{e}\times\langle\hat{\boldsymbol{s}}\rangle. \tag{1.4}$$

This equation is khown as a rigid-body spin precession equation, and it has a solution:

$$\langle\hat{\boldsymbol{s}}\rangle_t = \boldsymbol{e}\cdot\langle\hat{\boldsymbol{s}}\rangle_0\,\boldsymbol{e} + \sin(\Omega t)\,\boldsymbol{e}\times\langle\hat{\boldsymbol{s}}\rangle_0 - \cos(\Omega t)\,\boldsymbol{e}\times(\boldsymbol{e}\times\langle\hat{\boldsymbol{s}}\rangle_0). \tag{1.5}$$

One can see, that it means just a rotation around the axis $\boldsymbol{e}$ through an angle of Ωt. Therefore, the precession of a classical spin vector $\langle\hat{\boldsymbol{s}}\rangle$ describes an evolution of the spin state of a quantum system.

In accelerator physics, where the spin precession vector (operator) $\hat{\boldsymbol{\Omega}}$ can depend on the orbital dynamical variables $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$, the equation (1.3) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{s}}\rangle = \langle\hat{\boldsymbol{\Omega}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}})\times\hat{\boldsymbol{s}}\rangle. \tag{1.6}$$

Using the fact that $\hat{\boldsymbol{\Omega}}$ does not depend on $\hat{\boldsymbol{s}}$, one can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{s}}\rangle = \langle\hat{\boldsymbol{\Omega}}\rangle\times\langle\hat{\boldsymbol{s}}\rangle, \tag{1.7}$$

where the expectations are over the orbital and spin states, respectively. Note, that the expression (1.7) is still an equation of motion for a rigid-body spin rotation of a classical spin vector.

In general, however, is not simple to calculate the expectation value $\langle\hat{\boldsymbol{\Omega}}\rangle$. Therefore, to go further, one has to approximate that (semiclassical approximation):

$$\langle\hat{\boldsymbol{\Omega}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}})\rangle \simeq \boldsymbol{\Omega}(\langle\hat{\boldsymbol{x}}\rangle,\langle\hat{\boldsymbol{p}}\rangle), \tag{1.8}$$

Dropping out the averaging symbol for the classical variables:

$$\boldsymbol{x} \equiv \langle \hat{\boldsymbol{x}} \rangle, \qquad \boldsymbol{p} \equiv \langle \hat{\boldsymbol{p}} \rangle, \qquad \boldsymbol{s} \equiv \langle \hat{\boldsymbol{s}} \rangle, \tag{1.9}$$

one can write that $\boldsymbol{\Omega}_{sc} \equiv \boldsymbol{\Omega}(\boldsymbol{x}, \boldsymbol{p})$ for the semiclassical approximation. It is immideately follows that

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} \simeq \boldsymbol{\Omega}_{sc} \times \boldsymbol{s}. \tag{1.10}$$

Therefore, the evolution of the spin motion of a quantum system can be described by the classical spin vector $\boldsymbol{s}$ under the action of a semiclassical spin precession vector $\boldsymbol{\Omega}_{sc}$. In such approximation, this statement is true even if the spin precession vector depends on the orbital motion.

2. Semiclassical Approximation in Accelerator Physics

In spite of the fact that the semiclassical approximation is not a part of Ehrenfest's theorem, we still do need it to proceed further. Let us to discuss it in more details in this section.

Considering the case with a Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\boldsymbol{x}}). \tag{2.1}$$

One can immideately derive the second-order differential equation (Newtonian equation):

$$m\frac{\mathrm{d}^2\langle\hat{\boldsymbol{x}}\rangle}{\mathrm{d}t^2} = -\langle\boldsymbol{\nabla}_{\hat{\boldsymbol{x}}}V(\hat{\boldsymbol{x}})\rangle, \tag{2.2}$$

or two first-order differential equations

$$\frac{\mathrm{d}\langle\hat{\boldsymbol{x}}\rangle}{\mathrm{d}t} = \frac{\langle\hat{\boldsymbol{p}}\rangle}{m}, \qquad \frac{\mathrm{d}\langle\hat{\boldsymbol{p}}\rangle}{\mathrm{d}t} = -\langle\boldsymbol{\nabla}_{\hat{\boldsymbol{x}}}V(\hat{\boldsymbol{x}})\rangle, \tag{2.3}$$

which are describe the evolution of $\langle\hat{\boldsymbol{x}}\rangle$ and $\langle\hat{\boldsymbol{p}}\rangle$ as in original of the Ehrenfest's theorem. The application of the Ehrenfest theorem for the expectation value of the spin operator was first used by Bloch (1946). Moreover it is possible to apply it to the evolution of the expectation value of any quantum operator in modern physics.

Since a semiclassical description of the orbital trajectory is only valid if the spread in the coordinates and momenta are negligible, one can immideately see a fundamental limitation of the semiclassical approximation. One must require $\sigma_x \ll |\langle\hat{x}\rangle|$ and $\sigma_p \ll |\langle\hat{p}\rangle|$, where $\sigma_x^2 = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2$ and $\sigma_p^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2$ are standard deviations. To the extent that $\sigma_{\boldsymbol{x}}$ and $\sigma_{\boldsymbol{p}}$ are negligible, we can then approximate, for any function g,

$$\langle\hat{g}(\boldsymbol{x},\boldsymbol{p})\rangle \simeq g(\langle\hat{\boldsymbol{x}}\rangle, \langle\hat{\boldsymbol{p}}\rangle), \tag{2.4}$$

and

$$g_{sc} \equiv g(\boldsymbol{x},\boldsymbol{p}) = g(\langle\hat{\boldsymbol{x}}\rangle, \langle\hat{\boldsymbol{p}}\rangle). \tag{2.5}$$

It is clear, that a highly localized wavepacket can fulfill the semiclassical constraints. It may be considered as a "point" particle which evolves according to a (semi)classical equation of motion. If the wavepacket is composed of eigenstates of highly excited energy levels, the uncertainty in the energy is negligible relative to the average energy, and we can speak about a definite (semiclassical) energy, too; throughout this paper, one should interpret terms like "position", "momentum", "energy" and "angular momentum" in the semiclassical sense.

3. Spin Precession Equation

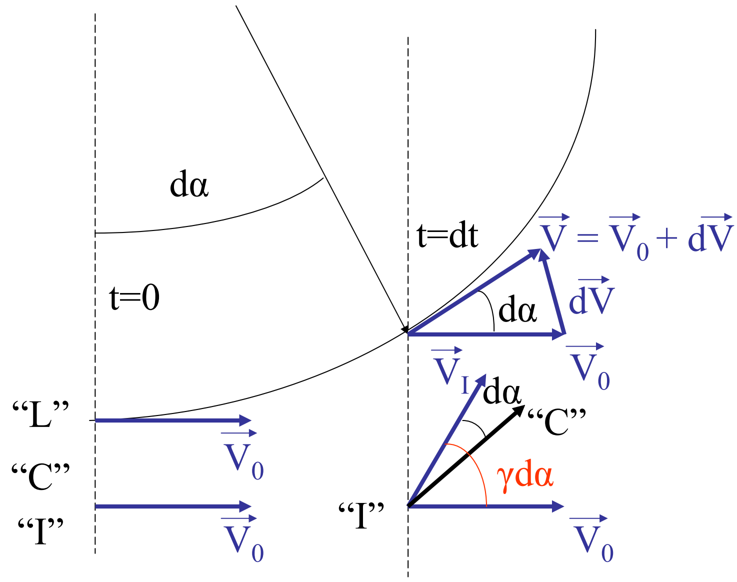

Figure 1. Coordinate systems and inertial frames for spin and orbital motion (The picture is taken from:*S. R. Mane, Yu. M. Shatunov and K. Yokoya "Spin-polarized charged particle beams in high-energy accelerators"*).

Let us to derive the spin precession equation using the discussed semiclassical approximation and the classical spin model as given "a priori". At this point one should specify the form of the spin precession vector in terms of laboratory electric and magnetic fields. For a nonrelativistic spin, the Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B} = -\frac{ge}{2mc}\boldsymbol{s} \cdot \boldsymbol{B}. \tag{3.1}$$

where all the symbols have their usual meanings, and the spin precession equation becomes

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{\mu} \times \boldsymbol{B} = -\frac{ge}{2mc}\boldsymbol{B} \times \boldsymbol{s}. \tag{3.2}$$

Further, we will generalize this result to the reletivistic case as it was given by Shatunov (2001) using just the Lorentz transformations, which derivation follows the approach of Thomas (1927).

Consider a particle moving with relativistic velocity $\boldsymbol{v}$ under the actions of laboratory frame electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively, along a trajectory determined by the Lorentz force equation, with $\boldsymbol{\beta} = \boldsymbol{v}/c$:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\,(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}), \tag{3.3}$$

from which it follows that

$$\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t} = \frac{e}{mc\gamma}\,(\boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{\beta}\boldsymbol{\beta} + \boldsymbol{\beta} \times \boldsymbol{B}). \tag{3.4}$$

Let us to define three different frames: laboratory frame "L", particle rest-frame "C" and an inertial frame "I" moving with the particle velocity $\boldsymbol{v}_{t=0}$ (figure 1). Also, let the coordinate origins of the frames C and I to coincide with the lab frame L at $t = 0$. We define that the Lorentz transformation from the lab frame L to the frames I and C at $t = 0$ be accomplished by a single boost, with a Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$. Note, that the spin $\boldsymbol{s}$ is defined for the "rest-frame" C.

It is otherwise possible to define another rest-frame C', defined as follows: we boost from L to an intermediate frame F using a boost with a velocity $\boldsymbol{v}_1$, which is arbitrary, and then perform a second boost with a velocity $\boldsymbol{v}_2$ (such that the under the combined Lorentz boosts $\mathcal{L}(\boldsymbol{v}_2)\mathcal{L}(\boldsymbol{v}_1)$ the final particle velocity is zero: this is the frame C'). However, C' does not coincide with C – which is arise from the structure of the Lorentz group. The (homogenous) Lorentz group has six generators: three rotations and three boosts, along the x, y and z axes, say. The rotations by themselves form a noncommuting subgroup $SO(3)$, since a combination of any two rotations is also a rotation. However, the boosts do not form a subgroup: a combination of two non-parallel Lorentz boosts is, in general, a single boost plus a spatial rotation. The "rest-frame" spin $\boldsymbol{s}$ is not well-defined unless we insist that the Lorentz transformation from the lab frame to the particle rest frame be accomplished by a single boost.

The particle will rotate through an angle

$$\mathrm{d}\boldsymbol{\alpha} = \frac{\boldsymbol{\beta} \times d\boldsymbol{\beta}}{\beta^2} = \frac{\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}}{\beta^2}\,\mathrm{d}t, \tag{3.5}$$

at an infinitesimal time $t = \mathrm{d}t$. Therefore, for the motion in a prescribed external electric and magnetic fields, we have

$$\frac{\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}}{\beta^2} = -\frac{e}{mc\gamma}\left[\boldsymbol{B}_\perp - \frac{\boldsymbol{\beta} \times \boldsymbol{E}}{\beta^2}\right], \tag{3.6}$$

and

$$\boldsymbol{B}_\parallel = \frac{\boldsymbol{B} \cdot \boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2}, \qquad \boldsymbol{B}_\perp = \boldsymbol{B} - \boldsymbol{B}_\perp = \frac{\boldsymbol{\beta} \times (\boldsymbol{B} \times \boldsymbol{\beta})}{\beta^2}, \tag{3.7}$$

to denote the components of the magnetic field which are parallel and perpendicular to the velocity $\boldsymbol{v}$, respectively. Under a Lorentz boost with the relativistic factor γ from the lab frame to the moving frames at $t = 0$, one can find

$$\boldsymbol{B}_C = \gamma\left[\boldsymbol{B}_\perp - \boldsymbol{\beta} \times \boldsymbol{E}\right] + \boldsymbol{B}_\parallel. \tag{3.8}$$

The spin changes in the proper time interval $\mathrm{d}\tau = \mathrm{d}t/\gamma$ according to (3.2), by

$$(\mathrm{d}\boldsymbol{s})_I = -\frac{ge}{mc}\,\boldsymbol{B}_C \times \boldsymbol{s}\,\mathrm{d}\tau. \tag{3.9}$$

To find the change in the rest frame, one should to note that the "C" frame itself rotates relative to the inertial frame, through an angle $d\boldsymbol{\phi}$. Subtracting this rotation we get

$$\mathrm{d}\boldsymbol{s} = (\mathrm{d}\boldsymbol{s})_I - \mathrm{d}\boldsymbol{\phi} \times \boldsymbol{s}. \tag{3.10}$$

Since at time $\mathrm{d}t$ the old rest frame (at $t = 0$) is oriented at an angle $-\mathrm{d}\boldsymbol{\alpha}$ relative to the new velocity $\boldsymbol{v} + \mathrm{d}\boldsymbol{v}$ and since in the moving inertial frame, both of these directions are rotating γ times faster than in the laboratory frame, one can find the angle $\mathrm{d}\boldsymbol{\phi}$:

$$\mathrm{d}\boldsymbol{\phi} = \gamma\mathrm{d}\boldsymbol{\alpha} - \mathrm{d}\boldsymbol{\alpha} = (\gamma - 1)\mathrm{d}\boldsymbol{\alpha} \tag{3.11}$$

Therefore,

$$\mathrm{d}\boldsymbol{s} = \Big[-\frac{e}{mc\gamma}\boldsymbol{B}_C \times \boldsymbol{s} - \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\times\boldsymbol{s}\Big]\,\mathrm{d}t. \tag{3.12}$$

Or in the other form

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\boldsymbol{s}. \tag{3.13}$$

The spin precession vector $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = -\boldsymbol{\mu}\cdot\frac{\boldsymbol{B}_C}{\gamma} + \boldsymbol{\omega}_T, \tag{3.14}$$

where $\boldsymbol{\omega}_T$ is the Thomas precession vector:

$$\boldsymbol{\omega}_T = -\frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = -\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}. \tag{3.15}$$

In terms of a Hamiltonian we have

$$H = -\boldsymbol{\mu}\cdot\frac{\boldsymbol{B}_C}{\gamma} + \boldsymbol{\omega}_T\cdot\boldsymbol{s}, \tag{3.16}$$

which contains two terms: the first is a magnetic dipole interaction with the rest-frame magnetic field (time-dilated by a factor γ) and the second is Thomas precession (Thomas 1927) due to the relativistic kinematics. Note that the Thomas precession term is independent of the electric and magnetic fields! It would exist even if the acceleration $\dot{\boldsymbol{v}}$ were due to gravitation or other non-electromagnetic causes. It is only that, in the cases of interest to us, the acceleration is also electromagnetic in origin, given by the Lorentz force.

The above form of the Hamiltonian is valid also for neutral particles possessing a magnetic moment (for example neutrons). One works directly with the magnetic moment $\boldsymbol{\mu}$ and does not introduce an electric charge etc, however, one can use g in conjunction with a Bohr magneton or a nuclear magneton. For charged particles one obtain

$$\boldsymbol{\Omega} = -\frac{e}{mc}\Big[\Big(a+\frac{1}{\gamma}\Big)\boldsymbol{B}_\perp + \frac{1+a}{\gamma}\boldsymbol{B}_\parallel - \Big(a+\frac{1}{\gamma+1}\Big)\boldsymbol{\beta}\times\boldsymbol{E}\Big], \tag{3.17}$$

where g can be decomposed into a sum of "normal" and "anomalous" parts, viz. $g = 2(1+a)$, where a is called the magnetic moment anomaly. The values of the magnetic moment anomalies of the various particles are shown in table 1. The table also shows the particle mass and the energy spacing of the so-called imperfection resonances.

One can note that we have expressed the fields in terms of components parallel and perpendicular to the velocity vector. An equivalent expression, which is also commonly used, and which does not decompose the fields into components, is

$$\boldsymbol{\Omega} = -\frac{e}{mc}\Big[\Big(a+\frac{1}{\gamma}\Big)\boldsymbol{B} - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}\cdot\boldsymbol{B}\boldsymbol{\beta} - \Big(a+\frac{1}{\gamma+1}\Big)\boldsymbol{\beta}\times\boldsymbol{E}\Big]. \tag{3.18}$$

Table 1. Magnetic moment anomalies of particles referenced in this article. The value for the muon is taken from Bennett *et al* (2004). The mass in MeV and the energy spacing of imperfection resonances is also shown.

Symbol	Anomaly $a = \frac{1}{2}(g-2)$	Accuracy	Mass (MeV)	$\Delta E = mc^2/a$ (MeV)
$e^{\pm}$	$1.1596521859 \cdot 10^{-3}$	$\pm 3.8 \cdot 10^{-12}$	$510.9989 \cdot 10^{-3}$	440.65
$\mu^{\pm}$	$1.1659208 \cdot 10^{-3}$	$\pm 6.0 \cdot 10^{-7}$	105.658	90622.24
p	1.792847351	$\pm 2.8 \cdot 10^{-8}$	938.272	523.34
d	-0.1429878	$\pm 5.0 \cdot 10^{-7}$	1875.613	13117.30

4. Breif analysys of the BMT equation and conclusions.

- The first point to note (and most important) is that $\boldsymbol{s}$ is expressed in the rest frame, while the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ are the fields in the laboratory frame. This parametrization is natural since the spin is an intrinsic property of the particle and is only truly meaningful in the particle rest frame.
- The anomalous part of the magnetic moment couples to the electric and magnetic fields in a different way. This lack of the symmetry in the coupling is due to the fact that the anomalous magnetic moment couples only to the rest-frame magnetic field (see (3.16)). The contribution of the normal magnetic dipole moment is mixed with the contribution from the Thomas precession.
- Wien filter: an important special case is that of a uniform magnetic field and an orthogonal electric field, such that the Lorentz force vanishes: $\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B} = 0$. For simplicity, suppose that $(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{\beta})$ form an orthogonal triad: the velocity is orthogonal to both the electric and magnetic fields. Then the acceleration of the particle vanishes: $\dot{\boldsymbol{\beta}} = 0$ and so the Thomas precession vanishes: the particle travels in a straight line, so all the Lorentz boosts commute. Then the Hamiltonian and spin precession vector are simply
$$H = -\boldsymbol{\mu} \cdot \frac{\boldsymbol{B}_C}{\gamma}, \qquad \boldsymbol{\Omega} = -\frac{ge}{mc}\,(\,\boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E}\,) = -\frac{ge}{\gamma^2 mc}\,\boldsymbol{B}. \tag{4.1}$$
Hence the orbit moves in a straight line but the spin precesses around the magnetic field. Such a device is called a Wien filter, and can be used to rotate the spin relative to the particle momentum. The effectiveness of a Wien filter falls off rapidly with energy, because of the factor of γ^2 in the denominator. It is also astigmatic, becauses it focuses the orbital motion but in only one plane.
- From (3.17), we see that for longitudinal and transverse magnetic fields of equal magnitude, the ratio of the effect on the spin precession due to the transverse and longitudinal magnetic fields is $(a\gamma + 1)/(a + 1)$. The value of $a\gamma$ greatly exceeds unity at the operating energies of most modern high-energy particle accelerators. Even in accelerators of a few GeV energy, $a\gamma \simeq 2-5$, while $a\gamma \simeq 70$ in HERA at 30

GeV, and $G\gamma \simeq 200$ in RHIC at 100 GeV. Recall that for protons it is conventional to write $G = (g-2)/2$. Hence transverse magnetic fields have a greater effect on the spin precession than do longitudinal magnetic fields. The ratio increases proportionately with energy.

- In the important case of a uniform vertical magnetic field, with no electric field, the orbital revolution frequency for horizontal circular motion is

$$\boldsymbol{\omega}_{rev} = -\frac{e}{\gamma mc}\boldsymbol{B}, \tag{4.2}$$

whereas the spin precession frequency is

$$\boldsymbol{\Omega} = -\frac{e}{mc}\left(a + \frac{1}{\gamma}\right)\boldsymbol{B} = (a\gamma + 1)\,\boldsymbol{\omega}_{rev}. \tag{4.3}$$

The ratio of the spin precession frequency to the orbital revolution frequency is called the spin tune. For a planar ring the value of the spin tune is

$$\nu_{\mathrm{spin}} = a\gamma, \tag{4.4}$$

where unity is subtracted because in the "accelerator frame" the spin precession is measured relative to the orbit, hence $\nu_{\mathrm{spin}} = (\Omega - \omega_{rev})/\omega_{rev}$. Equation (4.4) is simultaneously the simplest and most important equation in all of the spin manipulations. Note, that setting $\Delta(a\gamma) = 1$, i.e. $\Delta E = mc^2/a$, gives (see table 1)

$$(\Delta E)_e \simeq 440.65 \quad \mathrm{MeV}, \qquad (\Delta E)_p \simeq 523.34 \quad \mathrm{MeV}. \tag{4.5}$$

Hence $a\gamma$ or $G\gamma$ increases by one unit for an increase in the beam energy of 0.44 GeV (electrons) or 0.523 GeV (protons). The condition $a\gamma \gg 1$ or $G\gamma \gg 1$ is therefore satisfied by most accelerators, though not all. AmPS, SHR and VEPP-2M operate(d) at around $a\gamma = 1 - 2$ and the IUCF Cooler ran at $G\gamma \simeq 2$. LEP operated at $a\gamma = 100 - 200$, while RHIC at top energy will operate at $G\gamma \simeq 450$.

References

Bargmann V, Michel L and Telegdi V L 1959 *Phys. Rev. Lett.* **2** 435–6
Derbenev Ya S and Kondratenko A M 1971 *Sov. Phys.–JETP* **33** 658–64
Derbenev Ya S and Kondratenko A M 1972 *Sov. Phys.–JETP* **35** 230–6
Derbenev Ya S and Kondratenko A M 1973 *Sov. Phys.–JETP* **37** 968–73
Derbenev Ya S and Kondratenko A M 1976 *Sov. Phys. Dokl.* **20** 562
Derbenev Ya S, Kondratenko A M, Skrinsky A N and Shatunov Yu M 1977 *Proceedings of the 1977 Particle Accelerator Conference, vol. 2* (New York:IEEE) p 75 (in Russian)
Derbenev Ya S and Kondratenko A M 1978 *Sov. Phys. Dokl.* **24** 366–8
Derbenev Ya S, Kondratenko A M and Skrinsky A N 1979c *Particle Accelerators* **9** 247–66
Ehrenfest P 1927 *Z. Phys.* **45** 455
Jackson J D 1998 *Classical Electrodynamics, 3^{rd} ed* (New York:Wiley)
Shatunov Yu M 1969 unpublished
Shatunov Yu M 2001 *American Institute of Physics Conf. Proc.* **592** 279–316
Skrinskii A N and Shatunov Yu M 1989 *Sov. Phys. Uspekhi* **32** 548–54
Thomas L H 1927 *Philos. Mag.* **3** 1–22