Decay Suppresion in Degenerate Fermions

Hillel Wayne March 9, 2011

1 Intro

One of the fundamental tenets of identical is that of the Pauli exclusion principle. For a system with a given set of eigenkets, only one fermion may be defined by a given eigenket at one time. Otherwise the two fermion wavefunctions would destructively interfere and cancel both out. This property is generally used to explain degeneracy pressure and band gaps in solids, becoming a highly important topic in chemistry and solid state physics.

When combined with the ideas of conservation of energy, an interesting application comes up. Since any sort of particle decay creates new particles, some of these would be subject to exclusion principles. It then becomes possible to use the antisymmetric properties of the fermions to make the only available kets exceed the maximum energy of the decayed particle. In order to preserve conservation of energy the particle will be forced to remain in the unstable state. We will calculate the conditions under which this happens and attempt to find a practical way of testing it.

2 General Case

2.1 Energy Levels

Assume a cubical infinite potential well of length **L**, such that for a given particle of mass m the energy levels and wave function are respectively defined as

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \tag{1}$$

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{8}{L^3}} \sin\left(\frac{n_x\pi}{L}x\right) \sin\left(\frac{n_y\pi}{L}y\right) \sin\left(\frac{n_z\pi}{L}z\right) \tag{2}$$

We can consider the particle as lying in n-space as defined by the vector $||\vec{n}||^2 = \frac{n_x^2 + n_y^2 + n_z^2}{l^2} = \eta^2$. A degenerate fermion gas will first occupy the points of lowest magnitude n such that E_{tot} is minimized. For a system with many particles (such as the ones under consideration) the occupied points approximate an octant of a sphere. Let g be the number of fermions that can occupy a given point at once.[1]. In the n-space every available state occupies a volume $\frac{1}{V^3}$. The highest energy state of the gas is determined by η_m, or the radius of the octant. A short calculation gives

[1]For example, for electrons $g = 2$ (one $\uparrow$ and one $\downarrow$)

$$\frac{1}{8}(\frac{4}{3}\pi\eta_m^3) = \frac{N}{g}\frac{1}{V}$$

$$\eta_m = (6\rho\pi)^{1/3} \tag{3}$$

Where $\rho \equiv \frac{N}{gV} = \frac{N}{gL^3}$. Substitution into the energy equation gives

$$E_m = \frac{\hbar^2\pi^2}{2m}\eta_m^2 = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{N}{g}\right)^{2/3} \tag{4}$$

Where E_m is the fermi energy. [3]

2.2 Decays

We will preface that a, b, c are arbitrary particles. Let a be a particle that decays via $a \to b + c$, where b is a fermion. In all calculations the value c is meaningless when it refers to the particle. Therefore we can assume without confusion that a quantity subscript c refers to a quantity of the particle c, while c by itself refers specifically to the speed of light. In the rest frame of a we have

$$p_b = p_c \tag{5}$$

$$\frac{p_b^2}{2m_b} + m_bc^2 + \frac{p_c^2}{2m_c} + m_cc^2 + \Delta_b + \Delta_c \leq m_ac^2 + \Delta_a \tag{6}$$

Where Δ_i is the decay-energy uncertainty of particle i. in the limit of $\Delta_i << m_ic^2$ these terms can be ignored and (6) becomes an equality. We see that (6) becomes

$$\frac{1}{2}(\frac{1}{m_b} + \frac{1}{m_c})p_b^2 + (m_b + m_c)c^2 = m_ac^2$$

$$p_b^2 = 2(m_a - m_b - m_c)m'c^2$$

Where $m' \equiv \frac{m_am_b}{m_a+m_b}$ is the reduced mass of the two particles. Using $E_b = KE_b + ME_b$, ME the mass energy, we get

$$E_b = \left(\frac{(m_a - m_b - m_c)}{1 + \frac{m_b}{m_c}} + m_b\right)c^2 \tag{7}$$

in the case where $\Delta_a = \Delta_b = \Delta_c = 0$. Note that most cases these are not the only decay products and massless particles are also generated. Let ϵ be the total energy of the massless particles and $\vec{\rho}$ the total momentum. In this case we have to be far more careful about our coordinate system. It is not $p_b = p_c$ but $p_b + p_c = 0$, which leads to $p_b = -p_c$. Given we were

only dealing with squares of the momentum the negative sign was irrelevant. Here however we will have cross terms, and have to redefine momentum to account for that:

$$\vec{p}_b + \vec{p}_c + \vec{\rho} = 0, \vec{p}_c = -\vec{p}_b - \rho$$

$$\frac{1}{2}(\frac{p_b^2}{m_b} + \frac{p_c^2}{m_c}) - (m_a - m_b - m_c)c^2 + \epsilon = 0$$

We define the angle between $\vec{p}_b$ and $\vec{\rho}$ as θ, so that $\vec{p}_b \cdot \vec{\rho} = p_b \rho \cos\theta$. We also let $\varphi \equiv [(m_a - m_b - m_c)c^2 - \epsilon]/2$ for the sake of readability.

$$\frac{p_b^2}{2m'} + \frac{\rho^2}{2m_c} + \frac{p_b \rho \cos\theta}{2m_c} - \frac{\varphi}{2} = 0$$

$$p_b = \frac{-\rho\cos\theta/m_c \pm \sqrt{\left(\rho\cos\theta/m_c\right)^2 - 4\rho^2/m_c m' + 4\varphi/m'}}{2m'} \tag{8}$$

p_b is clearly maximized when we choose to add the descriminant and let $\theta = \pi/2$. Assuming this we note that p_b^2 will have the terms $\left(\frac{\rho}{m' m_c}\right)^2 - \frac{\rho^2}{m'^3 m_c}$, which simplifies to

$$(m_c^{-1} - m'^{-1})\frac{\rho^2}{m'^2 m_c}$$

Therefore the dependence on ρ will be negative when $m_c > m'$. Since m' denotes the average mass, we see that either $m_b \leq m' \leq m_c$ or $m_c \leq m' \leq m_b$. Therefore the momentum of the massless particles will raise the suppression energy if $m_b > m_c$ and lower it if $m_b < m_c$. In this case we leave the energy in the form

$$E_b = \frac{p_b^2}{2m_b} + m_b c^2 \tag{9}$$

For the sake of simplicity. Calculating E_b takes the form of calculating φ from ϵ and the masses of all involved particles followed by p_b^2 via equation (8). Note that uncertainty is neglected, and $\Delta p = \frac{\hbar}{2L}$, which macroscoping boxes negligable.

2.3 Suppression

Looking at (7) and (4) we see that the energy of a decay daughter product is dependent only on the masses involved, while the fermi energy is also dependent on the number of particles in the the system. Therefore one can arrange that the daughter particle will have less energy than the fermi energy. This happens when $E_b \leq E_m$, or

$$\frac{p_b^2}{2m_b} + m_b c^2 \leq \frac{\hbar^2 \pi^2}{2m_b L^2} \left(\frac{N_b}{g}\right)^{2/3}$$

However, this system dramatically overestimates N_b. The rest mass energy of the daughter particle cannot actually be applied to overcoming the fermi energy, as the FE is determined by the Hamiltonian of the particle. ME has remain in the form of mass. Therefore we must replace E_b with E_b', where $E_b' = KE_b$.

Solving in terms of N_b, we have

$$N_b \geq g \left(\frac{p_b L}{\hbar \pi}\right)^3 \tag{10}$$

If a box filled with N b particles and a single a particle, then when the a particle decays the daughter b must go to an unoccupied state in the fermi sphere. When the conditions in (10) are satisfied then there will be no open states for the particle to occupy. Therefore the a particle will be unable to decay.

2.4 Effects of Temperature

An important element neglected so far is that of temperature. Heat energy is an additional source of energy for the a particle which may end up giving its daughter b enough energy to overcome the Fermi energy. For a degenerate fermi gas the thermal energy is

$$E = E_0 + E_t = E_0 + \frac{1}{2}\delta T^2 (V^2 N)^{1/3} \tag{11}$$

Where $\delta = (\frac{\pi}{3})^{2/3} \frac{m}{\hbar^2}$ [4] (Landau 1959). In the worst case scenario all of this additional energy is directly applied to the a particle. We have $\frac{\partial E_t}{\partial N} \propto N^{-2/3}$. However, for the fermi energy $\frac{\partial E_f}{\partial N} \propto N^{-1/3}$. Therefore we see that for sufficiently low temperatures, the fermi energy increases faster than the thermal energy of a particle. For $N >> 1$ (ie most cases) the thermal energy of a particle can be reasonably neglected. However, this is only applicable for very low temperatures. In the classical limit the fermi gas will begin to obey Boltzmann statistics and $E_t = \frac{1}{2} k_b N T$, $\frac{\partial E_t}{\partial N} \not\propto N$ and thermal energy will very quickly exceed the fermi energy. Therefore we can conclude that decay suppression is only viable in cases of very low temperature. For degenerate fermi gases "low temperature" happens when $k_b T << E_f$ (Landau 1959).

There is one more case we need to consider. It may be true that the thermal energy can be lower than the fermi energy while still overcoming suppression. It is possible that with a lower temperature, we can raise one particle in the fermi sphere to the fermi edge while pushing the new fermion into the newly unoccupied state. However, in this case we raise one particle from some $E_0 \to E_f$ and another from $0 \to E_0$, leading to the same case as simply raising one particle from the ground state to the fermi energy. The two are equivalent.

2.5 Example: Beta Decay

As an example we take the decay ${}^{21}_{11}Na \rightarrow^{21}_{10} Ne + e^+ + v_e$. [1] Neon-21 and e^+ are both fermions and can force Sodium-21 suppression. We can analyze both suppression in both cases. $m' = 5.486 \times 10^{-4}$ amu, [5] $(m_a - m_b - m_c)c^2 = 2.525KeV$, and $\epsilon = 2.4MeV$.[2] [2] Because of the very low mass of the neutrino we can approximate $\rho = \epsilon/c$. This gives $\rho_b = .084eV$. Since $\rho << \varphi$ we can assume to first order $\rho \approx 0$, which means $p_{Na} = p_{e^+} = 5.21 * 10^{-20} J \cdot m/s$. Both Neon-21 and positron modes of suppression require the same number of particles neglecting degeneracy. Since Neon-21 has a completely filled electron shell we have $g = 1$, while for e^+ $g = 2$. This gives $N = 4 \times 10^{42}$ and $N = 8 \times 10^42$ respectively for a one meter box. As will be noted in the next section, this means practically suppressing beta decay is unfeasible.

3 Gamma Decay and Electron Transitions

As a numerical example we take Technetium-98m, the nuclear isomer Technetium-98. It decays via the process ${}^{98m}Tc \rightarrow^{98} Tc + \gamma$ with a half-life of approximately 15 microseconds. [1] ${}^{98}Tc$ is a fermion and therefore can undergo decay suppression. Technetium is an especially simple case to analyze because γ is massless and there is no m_c. This means that the kinetic energy of the daughter ${}^{98}Tc$ is very low and we can approximately say that $\epsilon \approx (m_a - m_b)c^2$. This means $Q \approx 0$. In this specific circumstance we cannot the derived quadratic formula for p_b. Instead we can simply say $p_b = \rho$. Therefore, for technetium,

$$E' = \frac{\rho^2}{2m_b}$$

In this case $m_b = 97.91$ amu. [5] Technetium has five unpaired electrons and therefore $g = 2$, corresponding to the cases $\uparrow\uparrow\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow\downarrow\downarrow$. For photons $\rho = \epsilon/c = 90$ keV $/c$. [1] Under this, we have

$$N_b = 2\left(\frac{\epsilon L}{\hbar c \pi}\right)^3 = 6 \times 10^{33} L^3 m^{-3} \tag{12}$$

For $L = 1m$ This is approximately 10^{10} moles of technetium, corresponding to approximately three times the weight of the Empire State Building. Since technetium has a volume of approximately $1.23 \times 10^{-29} m^3$ it becomes apparent that the combined volume of technetium would greatly exceed the volume of the container. Even assuming ${}^{98}Tc$ was a point particle under the ideal gas law the pressure of on the walls of the box would be well over the highest pressures ever acheived in a laboratory.

However, this case does have one interesting consequence. For gamma decay the suppression number only depends on the energy of the emitted photon. We propose to extend 'decay' from general particle decay to all forms of particle transition. Take a deuterium atom

[2] The neutrino energy is actually for thorium beta decay and dramatically underestimates the suppression number. This makes suppression even less feasible in reality.

^2H. With one proton-electron pair and one neutron deuterium is a fermion. If we neglect fine structure the $2s \to 1s$ and $2p \to 1s$ transitions release approximately 10 eV. Given a system of ^2H in the ground state and a smaller number of ^2H in the first excited state ($^2H_+$) then the suppression number is only 8.4×10^{21}, corresponding to approximately 27.9 milligrams of matter. This corresponds to a fermi energy of 2.69×10^{-8} eV, which has a degeneracy temperature of 0.3 mK. While low, this is impossible to test. The degeneracy temperature can be raised by increasing E_f, either by increasing the number of particles in the box or reducing the volume. Changing the sides of the box to 1 cm raises the temperature to 3 K, but it has the unfortunate side effect of raising the pressure to over three atmospheres.

3.1 The One Dimensional Case

All of the prior calculations assume that the particles occupy a volume, in that they are confined to a box. However, the one dimensional case of confining them to a length is considerably different. In this case E_m is defined as

$$E_m = \frac{\hbar^2 \pi^2}{2gmL^2} N^2 \tag{13}$$

Decay suppression again occurs when $E_m \geq E_b$, or

$$N_b \geq \sqrt{g} \frac{p_b L}{\hbar \pi} \tag{14}$$

In this situation for ^{98m}Tc suppression occurs around 10^{11} particles for a line one meter long. $^2H_+$ transition suppression occurs around 10^7 particles. Furthermore $\frac{\partial E_f}{\partial N} \propto 2N$ such that thermal energy becomes negligible even for very low N.

The one-dimensional square well is an acceptable approximation if two dimensions of the box are smaller than the de Broglie wavelength of the fermions. Since the highest energy fermions will have the lowest wavelengths, we use the fermi energy to judge the minimum such wavelength. Given the wavelength as h/p_f and the relation $E_f = p_f^2/2m$, we have

$$w \leq \frac{h}{\sqrt{2mE_f}}$$

This width is independent of the length of the well. For $^2H_+$ the minimum width is under 120 nanometers. While interesting from a theoretical perspective, it is unlikely that one dimensional suppression has practical application. This is because of two reasons: one, building a box with this minimum width is technically unfeasible. Two, the degeneracy temperature is directly dependent to the fermi energy while the minimum width is inversely proportional.

4 Neutrino Suppression

As a final example of decay suppression, we can naively (and perhaps facetiously) apply it to neutrino emission. Solar emission of neutrinos can have energies up to 18 MeV. Assuming the universe is a topologically closed box with a side length of 93 billion light years, that neutrino degeneracy is fourfold, and that neutrino has a rest mass-energy of 1 eV, the suppression number for solar neutrino production is

$$N = 2.47 \times 10^{111}$$

Nuclear fusion is effectively suppressed after this point. This is, of course, assuming that the universe is topologically closed, there is no 'unobserved' universe past the light barrier, and that neutrinos are not ever absorbed. Even if all of these assumptions are true the energy stored in these neutrinos would exceed the mass-energy of the universe by a factor of 10^{30}. This will not happen.

5 Conclusion

Through the use of large fermi spheres it is possible to suppress decays and transitions that produce the "forbidden" fermions. While practically and/or theoretically impossible to test for particle and isomer decay, it is feasible to test on electric dipole transitions. This opens up the possibility of using decay suppression to design new lasers and storage systems. We have not investigated how to design an experiment to test transition suppression, which is outside the scope of this paper. Developing such an experiment would be the logical next step in studying the potential phenomenon.

References

[1] G. Audi, O. Bersillon, J. Blachot, and A. H. Wapstra. The evaluation of nuclear and decay properties. *Nuclear Physics A*, 729(1):3 – 128, 2003. The 2003 NUBASE and Atomic Mass Evaluations.

[2] T. Akari et. al. Experimental investigation of geologically produced antineutrinos with kamland. *Nature*.

[3] David Griffiths. *Introduction to Quantum Mechanics*. 2nd edition edition, 2005.

[4] L. D. Landau and E. M. Lifshitz. *Course of theoretical physics. Vol. 5: Statistical physics*. Translated from the Russian by J. B. Sykes and M. J. Kearsley. Second revised and enlarged edition. Pergamon Press, Oxford, 1968.

[5] M. Wieser and T. Coplen. Atomic weights of the elements 2009 (iupac technical report). 83(2):359–396, 2009.