WEAK MEASUREMENT IN QUANTUM MECHANICS

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1. Introduction

In the November 2010 issue of Physics Today, Yakir Aharonov, Sandu Popescu, and Jeff Tollaksen discuss [5] a topic that has been controversial since Aharonov, Albert, and Vaidman (AAV) proposed it in 1987 with a paper boldly titled "How the Result of a Measurement of a Component of the Spin of a Spin-1/2 Particle Can Turn Out to be 100." [1] The topic is that of Weak Measurement and Weak Values of observables.

At the core of quantum mechanics is the notion of measurement: a measurement of an observable collapses the system being measured into an eigenstate of the Hermitian operator corresponding to the observable, and yields the corresponding eigenvalue. I refer to this standard process of measurement as "strong" or "projective" measurement. If the system is in a linear combination of eigenstates before the measurement, then a non-destructive measurement (that is, a measurement which leaves the state vector unchanged) is impossible.

The idea of Weak Measurement is that, while a decrease in the strength of the interaction between the measuring device and the system decreases the precision of the measurement, it also shrinks the disturbance that the measurement creates in the system. If the interaction is made very weak, then the system will be negligibly disturbed by the measurement, and any value measured will be meaningless by itself. However, if a large number of measurements is made, then the average will converge to a value defined as the weak value of the operator being measured. This determination of an operator's weak value through a large number of trials on identically prepared systems

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is known as weak measurement.

The entire goal of weak measurement is to learn about the expectation value of an observable without greatly disturbing the system (say, with the wavefunction collapse inherent in strong measurement). Of course, that disturbance of the system is only a concern when the post-measurement state of the system remains relevant to the experiment. For this reason, Weak Measurement is usually discussed in the context of the following canonical experiment.

Consider a system prepared in the state $|a\rangle$ at time t_0, subjected to a weak measurement of operator O at time t, then finally subjected to a measurement of operator B at time t_1 ($t_0 < t < t_1$). If the final measurement yields the eigenvalue b corresponding to the state$|b\rangle$, then the measurement is recorded; else, it is discarded. The system is then said to have been "preselected" in $|a\rangle$ and "postselected" in $|b\rangle$. The mean of the recorded values converges to the weak value of O given the pre- and postselection.

2. Derivation of the Weak Value in a Pre- and Postselected State

AAV derive [1,2] the weak value o_w for an observable o by considering a measuring device weakly coupled to the system under consideration via the standard Von Neumann interaction Hamiltonian. They use a formalism which describes pre- (post-) selection with a ket (bra) state that propagates into the future (past). They then derive the following as the weak value of an observable O measured in a preselected state $|a\rangle$ (eigenstate of observable A) and postselected state $\langle b|$ (eigenstate of observable B):

$$o_w = \frac{\langle a|O|b\rangle}{\langle a|b\rangle} \tag{1}$$

Note that one may derive this formula even without AAV's time-symmetric view of pre- and postselection, as I will do later. There is some disagreement in the literature over how to interpret any imaginary part that the above formula might have. Some authors [8] call the real part "the weak value." In AAV's derivation of the weak value as the value indicated by the pointer of a measuring device weakly coupled to the system, the authors state that the real part of the weak value describes the effect of the weak measurement on the position of the pointer, while the imaginary part describes the effect of that measurement on the momentum of the pointer. Either way, it appears that we should take the real part to obtain the weak value of an observable that is measured.

$$\begin{aligned} o_{w,re} &= Re\left(\frac{\langle a|O|b\rangle}{\langle a|b\rangle}\right) \\ &= \frac{1}{|\langle a|b\rangle|^2} Re(\langle b|a\rangle\langle a|O|b\rangle) \end{aligned}$$

$$o_{w,re} = \frac{\langle b|a\rangle\langle a|O|b\rangle + \langle a|b\rangle\langle b|O|a\rangle}{2|\langle a|b\rangle|^2} \tag{2}$$

2.1. **Weak Value from Bayesian Estimation Theory.** Johansen [3], drawing on the work of Hall [4], derives the weak value with Bayesian Estimation Theory given the prior knowledge obtained by pre- and postselection. I adapt his derivation for preselection in a pure (non-mixed) state and use less confusing notation[1]. First one considers an operator O' that is an estimator for O. O' will yield an estimate of O subject to the constraint that, because it corresponds to a weak (ie, non-destructive) measurement, it must commute with the postselection operator B. The quadratic loss is a measure of the efficiency (ie, accuracy, or lack thereof) of the estimate: $L(O') = \langle (O - O')^2 \rangle$, where expectation values are taken in the preselected state $|a\rangle$.

$$L(O') = \langle O^2 \rangle + \langle O'^2 \rangle - \langle OO' + O'O \rangle \tag{3}$$

One may now show that the last term $\langle OO' + O'O \rangle$ can be rewritten in terms of an operator $O_{w,re}$ corresponding to the weak measurement, thus commuting with B, whose eigenvalue is the real part of the weak value (Equation 2). It can be written in this way because O' and B are simultaneously diagonalizable.

$$O_{w,re} = \sum_{b'} |b'\rangle\langle b'| \frac{\langle b'|a\rangle\langle a|O|b'\rangle + \langle a|b'\rangle\langle b'|O|a\rangle}{2|\langle a|b'\rangle|^2} \tag{4}$$

Evaluation of $2\langle O'O_{w,re}\rangle$ reveals it can be written as $\langle OO' + O'O \rangle$.

$$2\langle O'O_{w,re}\rangle = 2\langle a|O' \left(\sum_{b'} |b'\rangle\langle b'| \frac{\langle b'|a\rangle\langle a|O|b'\rangle + \langle a|b'\rangle\langle b'|O|a\rangle}{2|\langle a|b'\rangle|^2} \right) |a\rangle \tag{5}$$

$$2\langle O'O_{w,re}\rangle = \sum_{b'} \langle a|O'|b'\rangle\langle b'| \frac{\langle b'|a\rangle\langle a|O|b'\rangle + \langle a|b'\rangle\langle b'|O|a\rangle}{|\langle a|b'\rangle|^2} |a\rangle \tag{6}$$

The second term simplifies easily. For the first term, recall that O' commutes with B by assumption. Thus, in the $|b'\rangle$ state, O' has the eigenvalue I denote by $O'(b')$

$$2\langle O'O_{w,re}\rangle = \sum_{b'} O'(b')\langle a|O|b'\rangle\langle b'|a\rangle + \langle a|O'O|a\rangle \tag{7}$$

$$2\langle O'O_{w,re}\rangle = \sum_{b'} \langle a|OO'|b'\rangle\langle b'|a\rangle + \langle a|O'O|a\rangle \tag{8}$$

$$2\langle O'O_{w,re}\rangle = \langle OO' + O'O \rangle \tag{9}$$

With this result, we can rewrite the loss function as:

$$L(O') = \langle O^2 \rangle + \langle O'^2 \rangle - 2\langle O'O_{w,re}\rangle = \langle O^2 \rangle - \langle O^2_{w,re} \rangle - 2\langle (O_{w,re} - O')^2 \rangle \tag{10}$$

[1]Beware that Johansen's A is the operator to be weakly measured, and his preselection is given by the density matrix ρ.

This is clearly minimized when $O' = O_{w,re}$, which proves that the real part of the weak value provides the most most efficient estimator of the operator O when the pre- and postselection are taken into account.

2.2. Efficiency of the Weak Bayesian Estimator $O_{w,re}$. We have seen that the estimator $O_{w,re}$, which gives the weak value $o_{w.re}$, is the most effecient estimator of O given pre- and postselection? Just how efficient is it? It turns out that the quadratic loss function, the measure of the efficiency of the estimator, is just the square of the imaginary part of the weak value.

First consider the operator $|O_w|^2 = O_{w,re}^2 + O_{w,im}^2$. Again as O_w (and thus $|O_w|^2$) commutes with B, the two are simultaneously diagonalizable.

$$|O_w|^2 = \sum_b |b\rangle\langle b| \left| \frac{\langle a|O|b\rangle}{\langle a|b\rangle} \right|^2 \tag{11}$$

$$\begin{aligned}
\langle |O_w|^2 \rangle &= \langle a| \left(\sum_b |b\rangle\langle b| \left| \frac{\langle a|O|b\rangle}{\langle a|b\rangle} \right|^2 \right) |a\rangle \\
&= \sum_b \langle a|O|b\rangle\langle b|O|a\rangle \\
&= \langle O^2 \rangle
\end{aligned}$$

Now the loss function can by simplified:

$$\begin{aligned}
L(O_{w,re}) &= \langle O^2\rangle - \langle O_{w,re}^2\rangle \\
&= \langle |O_w|^2\rangle - \langle O_{w,re}^2\rangle \\
&= \langle O_{w,re}^2\rangle + \langle O_{w,im}^2\rangle - \langle O_{w,re}^2\rangle \\
&= \langle O_{w,im}^2\rangle
\end{aligned}$$

Thus we have found that the quadratic loss of the estimator $O_{w,re}$ is just given by the square of the imaginary part of the weak value. This means that if the weak value o_w (Equation 1) is real, then it is an *exact* estimate of O between pre- and postselection.

3. Legitimacy of Postselection

Is postselection a legitimate operation? We would like to be able to speak of a particle preselected in $|a\rangle$ and postselected in $|b\rangle$. By that we mean that the operator A is measured with eigenvalue a, then later the operator B is measured to have eigenvalue b. We would like to be able to ask what the result of a weak measurement during the other two measurements would have produced. If we wanted to ask about the result of a strong measurement, then considering the postselection is illegitimate because a strong measurement might disturb the wavefunction so as to change the measured eigenvalue of B. However weak measurements are not projective, and commute with

B, thus postselection is legitimate and does provides new information.

Postselection is implemented experimentally as follows: Consider a device which produces particles in the $|a\rangle$ state and shoots them at a device which weakly measures O. The particles are then subjected to a standard (strong) measurement of B. When $B = b$ is measured, the measured value of O is recorded, else, it is discarded. To consider the weak measurement of an operator O given postselection in $|b\rangle$, means to measure O, but only record its value if $B = b$.

4. Impossible Spin Measurements

Consider a spin 1/2 particle preselected the $|+\rangle \equiv |S_z, +\rangle$ state and postselected in $|S_x, +\rangle$. What is the result of a weak measurement of the operator $S_{\pi/4} \equiv (S_x+S_z)/\sqrt{2}$ made between pre- and postselection? Using Equation 1, the weak value s_w is:

$$s_w = \frac{\langle +|S_x + S_z|S_x, +\rangle/\sqrt{2}}{\langle +|S_x, +\rangle} = \sqrt{2}\frac{\hbar}{2} \qquad (12)$$

Further, as s_w is real, note that (per Section 2.2) it is an exact estimate of the eigenvalue of $S_{\pi/4}$ between pre- and postselection. Of course, no strong measurement could ever yield this value, as it is not possible for any spin eigenvalue of a spin 1/2 particle to be greater in magnitude that $\hbar/2$. However, it is possible to understand why this value arises. Between pre- and postselection, the particle is "effectively" in a simultaneous eigenstate of S_x and S_z. By this I mean that one would clearly find $\hbar/2$ if one measured S_z, because of the preselection. Further, if one measured S_x, one would also find $\hbar/2$, now due to the postselection (as discussed in Section 3). A measurement of $S_{\pi/4}$ requires either a simultaneous projective measurement of S_x and S_z (which is impossible), or weak measurements of the two operators, which can be accomplished. Thus the measured (now weak) eigenvalue of $S_{\pi/4}$ is simply $(\frac{\hbar}{2} + \frac{\hbar}{2})/\sqrt{2} = \sqrt{2}\frac{\hbar}{2}$.

One possible way of implementing such a weak spin measurement is with a large magnet, ie, a large number N of spin 1/2 particles [5]. A weak measurement could consist of bringing a small piece of iron nearby to measure the direction and magnitude of the magnetic field. Because the piece is small, it does not greatly change the magnetic field of the magnet being measured. At the expense of precision, one may implement a weak measurement of the overall spin of the magnet in any direction desired. Supposing the unlikely (but possible) scenario in which all spins are measured in $|+\rangle$ initially (the preselection), and later measured in $|S_x, +\rangle$ (the postselection), one could weakly measure $S_{\pi/4}$ on a large ensemble of such systems and one would find $\sqrt{2}\frac{N\hbar}{2}$. As discussed earlier, any particular value obtained from a weak measurement is, by itself, meaningless. Only the expectation value of that weak measurement, the so-called weak value, is well-defined.

5. Hardy's Paradox

Aharonov et. al. have proposed [6] that consideration of weak values provides an intuitive way to resolve Hardy's paradox. The paradox arises when one considers the double Mach-Zehnder Interferometer pictured in Figure 1. First consider a slightly different version in which the paths of the electron and positron do not cross. The electron and positron enter the interferometer at the points indicated in the figure,

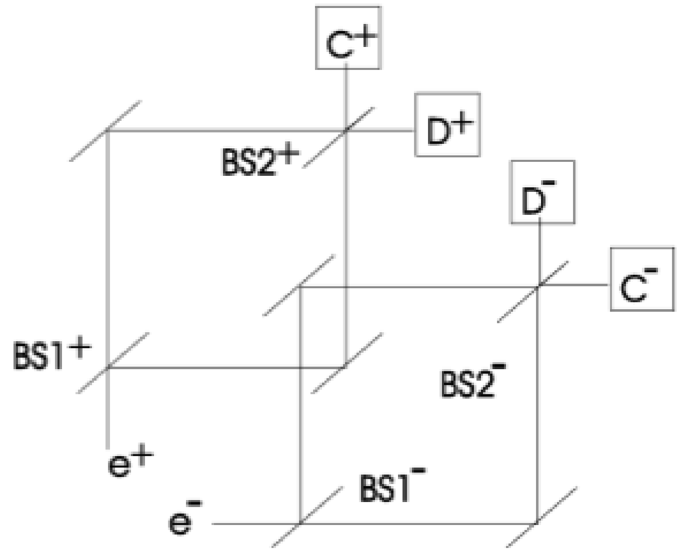

FIGURE 1. Hardy's gedanken setup. [6]

and each encounters 2 beamsplitters before arriving at detectors C+, D+, C-, or D-. Now suppose that the path lengths on each side of each beamsplitter are set such that the positron may only exit into C+, and the electron into C- (ie, the relative phases between the paths ensure that destructive interference prevents either of the particles from arriving at D+ or D-).

Now consider what happens when the paths of the particles cross, as in the figure. Such path crossing allows for the possibility of interference with the carefully crafted destructive interference that cancels the D+ and D- waves, making it possible for the particles to make it to the D detectors after all. Indeed suppose that both D- and D+ detect a particle. The supposed "paradox" arises from an attempted qualitative explanation of what must have happened. The D+ detection implies that the electron was in the overlapping arm of the interferometer, otherwise what could have caused the phase interference that should have prevented the positron from getting to D+. But then both particles were in the overlapping arm and should have annihilated, yielding no particle detections. All other lines of qualitative reasoning end similarly in contradiction.

It has been pointed out (See Ref [2] in [6]) that the basis of this paradox is counterfactual reasoning, that is, reasoning from evidence not obtained with measurements. We can infer where the electron or positron might have been, but have not actually measured the position of either. On that basis, some simply dismiss the paradox as one of the perils of counterfactual reasoning. And, of course, trying to measure (strongly) whether one of the particles is actually in the overlapping arm disturbs the experiment and removes the paradox [6].

But what about a *weak* measurement of one of the particles in the overlapping arm? Can the paradox be confirmed experimentally with a non-projective measurement? Interesting enough, it can. Following the argument in [6], let $|O_-\rangle$ and $|NO_-\rangle$ denote the electron states in the overlapping or non-overlapping arm of the electron path. After BS1-, the electron is in the $(|NO_-\rangle+|O_-\rangle)/\sqrt{2}$ state. By our assumption the lengths of the paths are such that, in the absence of interference by the other particle, constructive interference ensures that the electron reaches C-, while destructive interference prevents it from reaching D-. This means that C- only accepts $(|NO_-\rangle + |O_-\rangle)/\sqrt{2}$, while D- only accepts the orthogonal state $(|NO_-\rangle - |O_-\rangle)/\sqrt{2}$. The description of the positron side of the interferometer is the same but with +'s for the positron states on each of the positron arms.

After both particles are injected into the interferometer, the direct product state of the system is

$$(|NO_-\rangle + |O_-\rangle)(|NO_+\rangle + |O_+\rangle)/2 \tag{13}$$

Given that we want to consider the situation in which no annihilation is observed, we can project $|O_-\rangle|O_+\rangle$ out of the initial state above, yielding our preselected state $|a\rangle$

$$|a\rangle = (|NO_-\rangle|NO_+\rangle + |NO_-\rangle|O_+\rangle + |O_-\rangle|NO_+\rangle)/\sqrt{3} \tag{14}$$

We want to consider the situation where D- and D+ each detect a particle. Thus the final (ie, postselected) state is

$$|b\rangle = (|NO_-\rangle - |O_-\rangle)(|NO_-\rangle - |O_+\rangle)/2 \tag{15}$$

One may then compute the weak values of the single particle occupancy operators to find out the expected number of each particle present in each part of the interferometer between successful pre- and postselections, as well as the pair occupancy operators, which yield the expected number of particle pairs in the specified states.

$$\begin{aligned} N_{NO_+} &= |NO_+\rangle\langle NO_+| & N_{NO_-} &= |NO_-\rangle\langle NO_-| \\ N_{O_+} &= |O_+\rangle\langle O_+| & N_{O_-} &= |O_-\rangle\langle O_-| \end{aligned}$$

$$\begin{aligned} N_{NO_+,NO_-} &= N_{NO_+}N_{NO_-} & N_{NO_+,O_-} &= N_{NO_+}N_{O_-} \\ N_{O_+,NO_-} &= N_{O_+}N_{NO_-} & N_{O_+,O_-} &= N_{O_+}N_{O_-} \end{aligned}$$

Evaluation of the weak expectation values of these operators (using Equation 1) gives us of the results of weak (ie, non-projective) measurements of where each particle can be found between the pre- and postselection. Here I evaluate two of these operators, and state the results for the others. I denote the weak values of the particle occupancy operators with a superscript w.

$$N^w_{O_-} = \frac{\langle a|N_{O_-}|b\rangle}{\langle a|b\rangle} \tag{16}$$

$$N^w_{O_-} = \frac{\langle a|O_-\rangle\langle O_-|b\rangle}{\langle a|b\rangle} \tag{17}$$

$$N^w_{O_-} = \frac{\langle NO_+|(-|NO_+\rangle + |O_+\rangle)}{1-1-1} = 1 \tag{18}$$

One may evaluate one of the pair occupancy operators as follows.

$$N^w_{O_-,O+} = \frac{\langle a|N_{O_-,O_+}|b\rangle}{\langle a|b\rangle} \tag{19}$$

$$N^w_{O_-,O_+} = \frac{\langle a|O_-\rangle\langle O_-|O_+\rangle\langle O_+|b\rangle}{\langle a|b\rangle} \tag{20}$$

$$N^w_{O_-,O_+} = \frac{\langle NO_+|\langle O_-|O_+\rangle(-|NO_-\rangle + |O_-\rangle)}{-1} = 0 \tag{21}$$

The two bras on the left side of the numerator operate on different Hilbert spaces, so they commute through each other, making the numerator proportional to the term $\langle NO_+|O_+\rangle$, which is zero. The weak values of the other occupancy operators are evaluated similarly:

$$\begin{aligned} N^w_{NO_+} &= 0 & N^w_{NO_-} &= 0 \\ N^w_{O_+} &= 1 & N_{O_-} &= 1 \end{aligned}$$

$$\begin{aligned} N^w_{NO_+,NO_-} &= -1 & N^w_{NO_+,O_-} &= 1 \\ N^w_{O_+,NO_-} &= 1 & N^w_{O_+,O_-} &= 0 \end{aligned}$$

Remarkably, these weak occupation values agree with the above intuitive understanding of what must have happened for both the D- and D+ detectors to register counts. If the particle positions were measured weakly, each particle would be found in the overlapping arm of its side of the interferometer, otherwise nothing would interfere with the other particle and prevent it from reaching its C detector. However, one would not measure both particles in their overlapping arms because our assumption is that there is no annihilation.

The strangest weak value here is that of $N^w_{NO_+,NO_-} = -1$. Aharonov et al claim that this result preserves the number of particles. For example, the electron, if weakly measured, would be found traversing the overlapping arm. There is only one electron, so there must have been no electron in the non-overlapping arm. Indeed the total number of electrons in the non-overlapping arm is given by $N^w_{NO_+,NO_-} + N^w_{O_+,NO_-} = 0$.

Experimental confirmation of these weak values has been reported [8,9] in studies of Hardy's Paradox.

6. Controversy over Weak Measurement

The controversy over Weak Measurement is centered on whether the weak value of an observable represents something physical about the system. Stephen Parrott has written several articles (such as [7]) arguing that the weak value formula (Equation

1) is actually *not* universal to all measuring setups, implying that weak values are not unique functions of preselection, postselection, and the observable to be weakly measured. He has criticized AAV's original derivation [1,2] as implicitly assuming a particular setup of a system weakly coupled to a certain measuring device. He provides several derivations of weak values, all with slightly different assumptions about the ways in which the measurement is carried out, which yield formulae different from Equation 1. Possibly AAV's derivation is in error, as it assumes a particular measuring setup. However, Johansen's derivation of the standard weak value (which I adapt in Section 2.1) is agnostic to the measurement procedure. It views the weak value as simply the best estimate one can make of on operator, taking both pre- and postselection into account. This presents strong evidence that the weak value represents a genuine physical value.

7. Uses of Weak Values

Aside from providing explanations of Hardy's Paradox and predicting Large Spin measurements, Weak Measurement has been shown to be useful as a form of amplification of a small value to make it measurable (See the second to last section of [5]). It is noted in that reference that measurement of Weak Values can be used to provide amplification without the added noise inherent to other experimental methods of signal amplification. It has been used (see same reference) to measure tiny displacements of laser beams and small mirror displacements in interferometers. If nothing else, weak measurement provides a way of probing intriguing aspects of quantum mechanics in a somewhat non-orthodox way, which has yielded valuable experimental insights.

8. References

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