Topological Quantum Computing

Jacob Colbert

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1 Introduction

Quantum computers have the ability to solve certain problems exponentially faster than the best known algorithms for classical computers. Examples include database searching, simulation of quantum systems, and factoring large numbers.[1][2][3] However, the nature of most quantum systems makes the states too fragile for efficient calculations to be done. Topological quantum computing avoids these problems by making the state immune to most types of noise.

In order to motivate the development of topological quantum computing we begin by discussing typical quantum computing its problems. We will then introduce the concept of anyons as a generalization of bosons and fermions. We then discuss the concept of non-abelian anyons and their composition through fusion channels. With this material covered we can then provide a general description of topological quantum computing and its advantages. We conclude by discussing the necessary conditions for finding non-abelian states in nature and the current promising candidates.

2 Typical Quantum Computing

2.1 What is Quantum Computing?

In order to understand the motivation for topological quantum computing we must first understand non-topological approaches to quantum computing and the problems they face. Quantum computing in its simplest form can be understood as initializing a quantum system in some Hilbert space, applying a series of unitary transformations on it, and then measuring the result. The series of unitary transformations is the program or the logical gates applied to the initial system in order to caluculate the desired quantity. Consider a two level system, $\psi = a \mid 0\rangle + b \mid 1\rangle$. In this example, the unitary transformation

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a the unitary transformation corresponding to the NOT-gate, as it takes an initially $\mid 0\rangle$ state to $\mid 1\rangle$ and vice-versa. Unitary transformations acting on higher dimensional Hilbert spaces corresponding to multiple qubits can be shown to be capable of performing all operations that classical circuits are capable of.

Quantum computing differs from classical computing however in that there are many unitary operators that have no classical analog. For example, one can apply the unitary operator

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

corresponding to a phase shift of one of the components. This reflects the additional information stored in quantum systems. While still being normalized to unity, the state can be in a linear combination of both $\mid 0\rangle$ and $\mid 1\rangle$, with any relative phase. This allows for parallelization of a computation along different paths, in the end taking the coherent sum of the result.[1]

In order to physically apply these gates to the initialized system one requires control over the Hamiltonian. From the standard result of quantum mechanics, the unitary time evolution operator satisfies

$$i\hbar\frac{\partial}{\partial t}U(t) = H(t)U(t)$$

[4]Thus with suitable control over the Hamiltonian of the system one is able to apply whichever unitary operation is required.

2.2 Quantum Error Correction

Error correction is more difficult in quantum computation than in classical computation. The errors are no longer discrete flips of $\mid 0\rangle$ into $\mid 1\rangle$, but can be continuous with the relative phase of the two states being altered. Additionally, the typical technique of keeping redundant copies of the data and comparing them throughout the computation will not work for quantum computations. Measuring the state of the qubit at an intermediate step in the computation will collapse the superposition of the state forcing it to be either $\mid 0\rangle$ or $\mid 1\rangle$, resulting in a loss of the relative phase information.

Despite these difficulties, quantum error correction algorithms have been found. One such algorithm stores the quantum information redundantly, so that the information of one logical qubit is stored in three qubits. That is, the state that used to be described as, $\psi = a \mid 0\rangle + b \mid 1\rangle$ is now described by $\psi = a \mid 000\rangle + b \mid 111\rangle$, with each number representing an individual qubit. Let us first consider only the problem of bit flips, rather than phase shifts. Now the information can be verified mid-computation by comparing the values of the three qubits that make up the logical qubit to each other without measuring the state of the whole system. If its found that there is an inconsistency between the three qubits, the qubit that does not agree with the others is flipped so that they all agree.[1] This model can be extended by noting that phase shifts in one basis correspond to a bit flip in the other basis.

With error correction algorithms it becomes possible to efficiently carry out quantum computation as long as the error rate is low enough. If the error rate is too high, it is possible that the error correction itself will contain errors, which will then have to be error corrected and so on. In order to avoid this problem a cutoff on the error rate is estimated to be between 10^{-4} and 10^{-6}; that is, it must be able to apply 10^4 to 10^6 operations without an error.[2] This bound is very strict for the fragile quantum systems used to do quantum computation.

In typical quantum computers there is the problem of decoherence of states. Decoherence of a state is when the qubit state interacts with the environment and the two become entangled. As a precise measurement of the environment is not possible, the information which is coupled to the environment is now lost. In addition to errors caused by storing quantum qubits there are problems due to the precision with which a quantum gate can be applied. A gate meant to shift the phase by $90°$ could accidentally shift by $90.01°$ resulting in an incorrect calculation.

These problems with the typical approaches to quantum computing motivate the ideas of topological quantum computing. Rather than trying to control these sources of error directly, topological quantum computing makes it so that the system is unaffected by small perturbations from the expected values. Put another way, "topological quantum computation does not try to make the system noiseless, but instead makes it deaf." [3]

3 Anyons

3.1 Abelian Anyons

Before defining topological quantum computation we must first explain the notion of the anyon. In standard treatments of elementary quantum mechanics particles are broken up into bosons and fermions depending on the statistics they satisfy. If the exchange of two identical particles leaves the state unchanged the particles are termed bosons, and if the state gains a negative sign the particles are termed fermions. It is postulated that all particles belong to one of these two classes.

This is the case in three dimensions, but this dichotomy does not necessarily hold in two dimensions. The reasons behind this are due to the topology of the spaces. We can sketch the difference as follows. Exchanging two particles twice is topologically equivalent to bringing one particle in a closed circle enclosing the other particle. In three dimensions however this is topologically equivalent to not moving either particle. The path taken

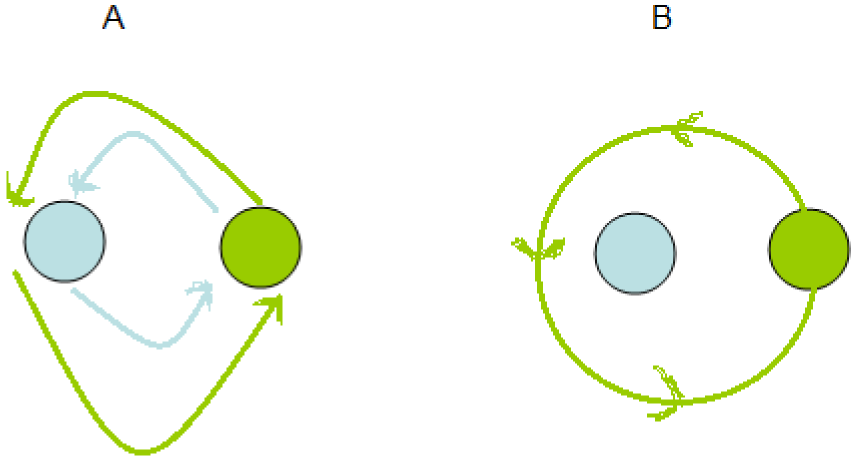

Figure 1: A. Two particles interchange position twice following the paths shown. B. The “topologically equivalent” case of one particle looping around the other.

encircling the other particle can be pulled up into the third dimension and then contracted so that the path becomes trivial. This concept is illustrated in figure 1. Thus we require that interchanging the particles twice results in no change to the system.

First let us consider a one dimensional subspace of the Hilbert Space, for example, if the particle is in a non-degenerate ground state of the system. If we denote the action of interchanging the two particles by the operator π, then the requirement of two interchanges returning the particle to its original state, requires that

$$\pi\pi = 1 = \pi\pi^{-1} = \pi\pi^{\dagger}$$

Thus π is its own inverse and the unitarity requirement of time evolution of quantum states requires that $\pi = \pi^{\dagger}$. So π is a real matrix. Furthermore, we get that the eigenvalues of π must all be 1 or -1. In our one dimensional case, this means that the matrix of π must just be 1 or -1, as we expect for bosons and fermions respectively.

This is not the case in two dimensions. With no extra dimension to pull the path of the traveling particle through as we did in two dimensions, there is a topological difference between the trivial path where both particles stay in place, and the one where they are interchanged twice. Thus we can no longer require that a single interchange multiply the state by 1 or -1, and now any unit modulus complex number $e^{i\theta}$ will work. The particles are then said to obey θ-statistics. These statistics are a generalization of the Bose-Einstein and Fermi-Dirac statistics obeyed by bosons and fermions. They correspond to $\theta = 0$ and $\theta = \pi$ respectively. These particles can have any phase factors and are thus called ”anyons”.

These ideas can be formalized using the concept of homotopy of curves from topology. The concept of homotopy is what we invoked above when talking about paths being ”topologically equivalent.” Formally, two paths, $f(t)$ and $g(t)$, in a space X are said to be homotopic if there exists a continuous function

$$F : [0,1] \times [0,1] \to X$$

where

$$F(t,0) = f(t) and F(t,1) = g(t)$$

This notion corresponds to deforming the path $f(t)$ continuously into the path $g(t)$ as the second coordinate ranges from 0 to 1. The set of curves which are homotopic to each other form an equivalence class, partitioning all curves in X. When considering braids of multiple particles, we do not allow these braids to self intersect, as this would imply that the particles are going through each other. That is, two sets of paths, $f_i(t)$ and $g_i(t)$, in the set $X - \bigcup_{j\neq i} f_j([0,1])$ and $X - \bigcup_{j\neq i} g_j([0,1])$ respectively are homotopic if there exists continuous

$$F_i : [0,1] \times [0,1] \to X - \bigcup g_j([0,1])$$

where again we require

$$F(t,0) = f_i(t) and F(t,1) = g_i(t)$$

[8] For technical reasons we sometimes consider the set of links formed by closing the braids into closed loops.

We now consider the world lines of n particles which are being interchanged, so that the set of final coordinates of the particles is the same as the inital set of coordinates, while not requiring each one be returned to its initial position. In 3+1 dimensions, denoting 3 spatial dimensions and a time dimension, the set of all homotopy classes of these world lines as described above has the group structure of S_n, the permutation group on n letters. We can see from the above argument that one is always able to deform the paths trivially by taking advantage of the third dimension. The only non triviality comes from the rearrangement of the initial and final coordinates. The group operation is taken by composing two interchanges of particles, thus producing the group S_n.

Now we need to see how the interchanges act on the system. This is the question of how the group is represented by linear transformations. If we first consider one dimensional representations of the symmetric group, we know from the representation theory of finite groups that there are only two possible actions of the group. One possible action is the trivial representation, $\rho(\sigma) = 1$, which takes each element of the permutation group to the identity transformation. Another possible action is the sign representation, $\rho(\sigma) = sign(\sigma)$, which takes the permutation to the sign of the permutation times the identity transformation. These two cases correspond to the case of bosons and fermions respectively. All higher order representations can be broken down into fermions and bosons, but this is beyond the scope of this paper.[5]

In 2+1-dimensions we have non-trivial braiding. As before it is not just the final permutation of the coordinates that matters, as the paths can no longer be un-braided in 2 dimensions. It now matters whether a particle is taken around another clockwise or counter-clockwise, for example. In this case the group formed by the homotopy classes is the braid group on n particles, B_n. The braid group is generated by clockwise switches of adjacent particles. That is the set of all σ_i where σ_i is the clockwise exchange of particles i and $i + 1$ generate the braid group. The properties of the braid group are shown in figure 2. The braid group is infinite unlike the finite symmetric group and thus has much

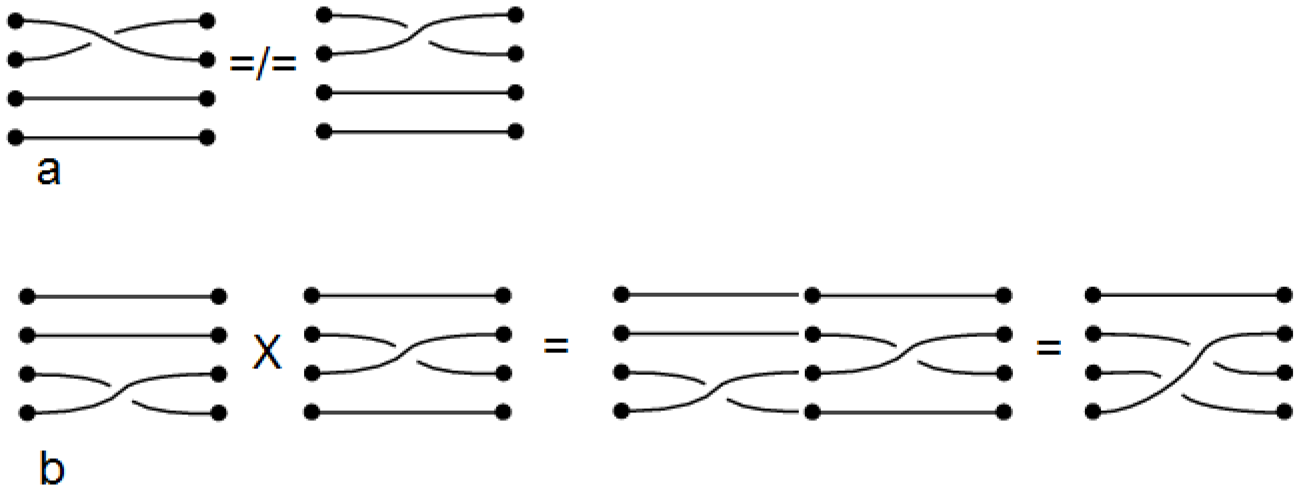

Figure 2: A. Clockwise and counter-clockwise exchange of particles are not equivalent as elements of the braid group. B. The group operation of the braid group is composing the braids to form a new braid.[7]

richer representations. The map, $\rho(\sigma_i) = \eta$, is a one dimensional representation for any $\eta \in \mathbb{C}$.[6] Along with requiring unitary or probability conservation we get that $\eta = e^{i\theta}$. Thus we have reproduced the concept of anyons in 2+1 dimensions using the formalism of representation theory.

3.2 Non-Abelian Anyons

Considering once again the problem of quantum computation we can see that systems in which interchanges of particles are represented by one dimensional representations will not provide a universal quantum computation. Rather than providing any unitary transformation required for a computation, these interchanges only multiply the entire state by a phase factor. We instead will look for systems in which interchanges are non-abelian representations. We note that it is necessary but not sufficient that the homotopy group of the interchanges be non-abelian, which it is, because otherwise

$$\rho(\sigma_1)\rho(\sigma_2) = \rho(\sigma_1\sigma_2) = \rho(\sigma_2)\rho(\sigma_1)$$

and the representation is necessarily abelian. As we saw above with one dimensional representations of non-abelian groups, this does not ensure a non-abelian representation and we must explicitly require that $\rho(\sigma_1)\rho(\sigma_2) \neq \rho(\sigma_2)\rho(\sigma_1)$ for some $\sigma_1, \sigma_2 \in B_n$

3.3 Fusion Channels

Let us now go back to the case of abelian anyons with θ-statistics. We now consider the composite of two anyons as a single anyon and see the statistics it obeys. Even if no actual bound states of the two anyons exist, for the topological purposes it is still acceptable to consider them as a composite.[1] Now braiding two of these composite anyons around each

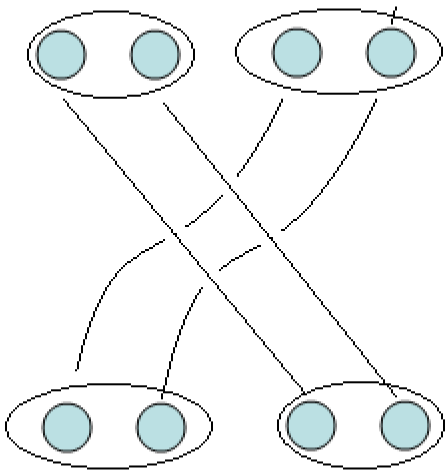

Figure 3: The braid of the composite anyons can be seen as 4 braids of the basic anyons.

other can alternatively be viewed as 4 braids of the basic anyons as shown in figure 3 Thus the composite anyons obey 4θ-statistics. General composites of n particles will obey $n^2\theta$-statistics. We can then compose these composite anyons. Composing two quasiparticles, one obeying $n^2\theta$ statistics the other obeying $m^2\theta$ statistics, we can compose them in order to get a quasiparticle obeying $(n+m)^2\theta$-statistics by generalizing the above picture. The combination of these anyons into a new anyon is called fusion and is denoted

$$n^2\theta \times m^2\theta = (n+m)^2\theta$$

where the particle is denoted by the statistics it obeys. In this way the anyon's statistics are a topological quantum number denoting how it braids with other particles.

Unlike the abelian case where the fusion of two anyons produces a new anyon with known statistics, the fusion of non-abelian anyons can be a linear combination of anyons satisfying two different statistics. This is analogous to the case of addition of angular momentum where the combination of two particles with known quantum numbers are then in a linear combination of states with different quantum numbers. We write that

$$\phi_a \times \phi_b = \sum_c N_{ab}^a \phi_c$$

which is analogous to the expansion of spins in terms of the Clebsch-Gordon coefficients. In the abelian case we know that $N_{ab}^c = \delta_{c(a+b)}$. That is, for each a and b there is one non-zero N_{ab}^c when $c = a + b$. In the non-abelian case we can guarantee that there is some a and b such that there are multiple c with $N_{ab}^c \neq 0$.[1]

Let us now choose a specific example. We will consider a system that has three different types of anyons, denoted 0, $\frac{1}{2}$, and 1. The fusion rules are give by

$$0 \times a = a$$

$$1 \times 1 = 0$$

$$1 \times \frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

This is similar to the addition of spin $\frac{1}{2}$ particles requiring the maximum spin to be 1. Now consider a system containing four $\frac{1}{2}$ particles, labeled 1, 2, 3, and 4. If we know them to have total topological quantum number 0, i.e. if they were created in pairs from the vacuum, than we know that any two pairs must fuse to the same anyon. If 1 and 2 fuse to 0 then 3 and 4 must fuse to 0 as well. Consequently the Hilbert space of states of the system is 2 dimensional. We have a state ψ_1 corresponding to both pairs being 1 and ψ_0 corresponding to both pairs being 0. However, we could alternatively group anyons 1 and 3 together. Then we have a basis ψ_0' and ψ_1' corresponding to this composite particle being either 0 or 1. Thus we must have a change of basis matrix, called the F matrix.

Lastly, we must see how the topological quantum number, or topological charge, is effected by braids of quasiparticles. Unfortunately the bulk of this material is beyond the scope of this paper and comes from results of conformal field theory. We can, however, note a few properties. The topological charge of the a composite quasiparticle can not be changed by braiding of internal anyons. Thus only multiplication by a phase is allowed by braids within the quasiparticle. The phase factor will depend on the particle types, and the set of phases are given in the R matrices.[1] Additionally, the braiding of an external anyon with one, but not all, of the anyons in a composite quasiparticle will change the species of the composite quasiparticle. A system of anyons is defined by its fusion channels, F matrices, and R matrices.

4 Topological Quantum Computing

We now have the necessary tools to see how topological quantum computing addresses the problems faced by conventional approaches. We first consider generally how computations are carried out. As mentioned before, a computation involves initialization of an initial state, the application of unitary transformations, and measurement of the final state.

Initialization of the system can occur in several ways. One way is the creation of pairs of quasiparticles from the vacuum. Just as angular momentum is conserved in pair production, we require that the pair have no total topological charge, i.e. the pair braids trivially with other quasiparticles. In this way the particles have a known initial value.[3] Another way to get a desired initial set up is to first perform a measurement so that the particles are in a known state, and then to evolve them with a unitary transformation to get the desired state.

To apply the desired unitary transformation one physically braids the quasiparticles to achieve the desired gate within the accuracy threshold desired. One way to physically braid the particles is to use a device like that in figure 4. In such a device there are quasiparticles in the bulk of the sample as well as edge currents of particles. Allowing an edge quasiparticle to tunnel through contact B in the figure would apply a non-trivial braid to the system. Where the quasiparticle tunnels can be controlled by applying voltage gates to the sample. The unitary transformation applied depends on the details of the system, and can be calculated from the F matricies and the R matricies defined above.[1] For a large class of non-abelian systems the set of available transformations is a dense subset of the set of all unitary transformations on the Hilbert space. These are

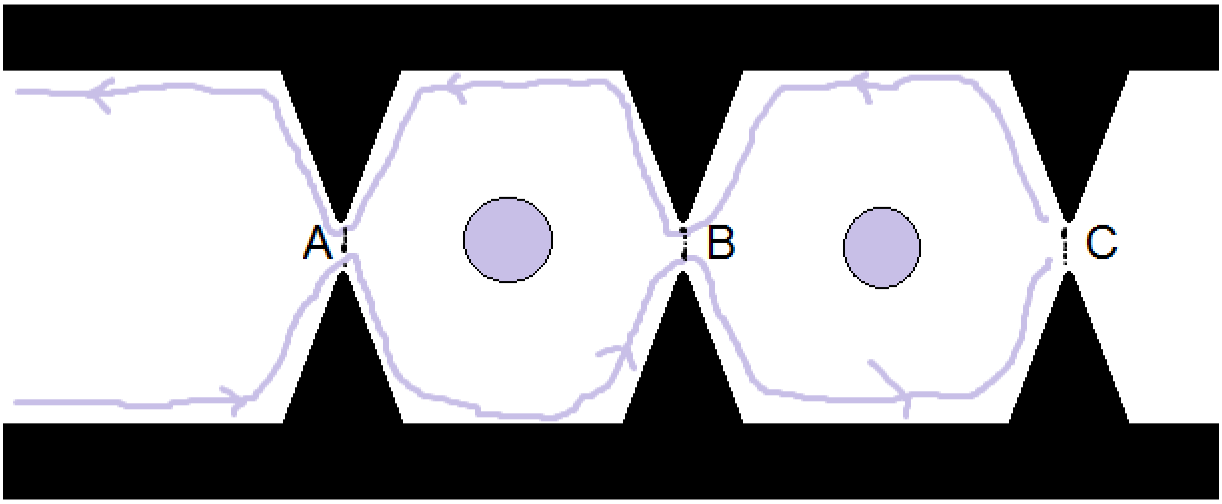

Figure 4: Diagram of device to measure and braid quasiparticles.

called universal quantum computers. That is, there is a braid that is arbitrarily close to any desired unitary transformation.

Measurement of the final state of the system is accomplished in a similar manner to the braiding. Again consider the device of figure 4. An edge current of particles is allowed to tunnel through contacts A and C. By measuring the interference pattern we are in effect measuring the state of the anyons localized within the loop as the phase change associated with surrounding the quasiparticles depends on the statistics of the composite quasiparticle. This method is similar to one approach experimentalists are using to search for non-abelian states in nature.

The advantage of topological quantum computing is in its robustness over local perturbations. The problems (discussed above) of the inaccuracy of the applied gates is removed because the transformations are now discrete, a particle either braids around another or it does not. There is still the problem of the braid not exactly matching the desired logical gate, but this error can be made arbitrarily small through the choice of the braid. The problem of perturbations affecting the storage of information is drastically decreased due to the topological robustness of the state. Only if the braiding of the quasiparticles is affected will the perturbation change the state. However, there is still the possibility of errors in this case. If a pair of anyons is created from the vacuum and braids around one of the quasiparticles in the qubit, the state of the qubit will change. Luckily, a large class of such events can be shown to have no effect on the state.[1]

5 Non-Abelian States in Nature

We describe here necessary conditions for the realization of non-abelian states in nature and then mention some promising candidates. The first condition necessary for non-abelian states is a degenerate ground state. In order for the exchange of quasiparticles to not just multiply by a phase there must be another linearly independent state for the quasiparticle. Additionally, the degeneracy must not be due to any spacial symmetry of the system. In this case local perturbations will destroy the degeneracy, and thus the system loses its topological robustness. Lastly, we need an energy gap between the ground state and the excited states. This energy gap provides protection to the ground state systems. Additionally, it defines the time scale, $T = 2\pi\hbar/\Delta E$, for what is meant by adiabatically braiding particles.[4]

While these conditions are all required to have non-abelian states, they are not sufficient. Even in these conditions the representation of the braid group that acts on the system could still be abelian. Experimental measurements of the system in question are necessary to see if non-abelian braiding statistics are actually present. There is currently no experimental evidence of non-abelian states in nature. The most promising system for non-abelian braiding statistics is the $\nu = \frac{5}{2}$ fractional quantum hall state.[2] The fractional quantum hall effect satisfies all of these conditions and theoretical work predicts that the $\nu = \frac{5}{2}$ plateau could have non-abelian statistics. Unfortunately, the work predicts that even if the state does have non-abelian states, they could not make the basis of a universal quantum computer, as the braiding is not rich enough to approximate any needed unitary transformation. However with just a few local (non-topological) quantum gates the state could be made into a universal quantum computer without sacrificing much of the accuracy. There is another candidate, the $\nu = \frac{12}{5}$ plateau, that it is predicted could be capable of universal quantum computing; however even less is known about this state.

6 Conclusion

We first considered the basic concepts of quantum computing. Using the problems it faces as motivation, we developed the notion of the anyon and its topological properties. From there we generalized this concept to non-abelian anyons and considered their fusion. We were finally able to develop the general notions of topological quantum computing and its advantages over standard approaches. We concluded by discussing the necessary conditions for non-abelian anyons to exist in nature and potential quantum systems in which they could arise.

7 Acknowledgements

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