Properties of Degenerated Fermi-Gas in Astrophysics

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Introduction

The degenerated fermi-gas is a dominated component in a highly dense region in Astronomy, such as the center of a white dwarf. The equation of state of the degenrated gas is therefore important to the evolution of a star. Different temperature and density correspond to different stages of evolution. I'm going to discuss the properties of degenerated gas and its application on Astronomy.

Part I

Degeneracy of Electrons

The density of electrons is described by Fermi-Dirac statstics as electrons have half-integral spin. For electron with momentum

$$|\overrightarrow{p}| = p$$

the density in the range (dp) can be descirbed by

$$n_e(p)dp = \frac{8\pi}{h^3}p^2 dp \times \left[\exp(\alpha + \frac{E}{kT}) + 1\right]^{-1}$$

Due to Pauli exclusion principle, two identical electrons can't occupy the same state, which means

$$P(p) = \left[\exp(\alpha + \frac{E}{kT}) + 1\right]^{-1}$$

can't be larger than 1. If the density of electrons increases, the electrons will be force to higher momentum states. The high momentum electrons contribute to the pressure, and the restriction on number density at each states is the souce of degeneracy pressure.

1 Complete degeneracy

We first consider a simple case which is $T \to 0$. When the density is high enough, all the electron states with energy less than a maximum energy are filled, and the density distribution becomes (α depends on density and temperature. It's a large negative number here.)

$$n_e(p)dp = \begin{cases} \frac{8\pi}{h^3}p^2dp & \text{for } p \prec p_0 \ , \ P(p)=1 \ \ \frac{E}{kT} \prec |\alpha| \\ 0 & \text{for } p \succ p_0 \ , \ P(p)=0 \ \ \frac{E}{kT} \succ |\alpha| \end{cases}$$

p_0 is also called Fermi momentum. The total number density of electron becomes

$$n_e = \int_0^{p_0} n_e(p)dp = \frac{8\pi}{3h^3}p_0^3 \ \text{ or } p_0 = h\left(\frac{3}{8\pi}n_e\right)^{\frac{1}{3}}$$

In the assumption of perfect gas, the pressure is calculated by $P = \frac{1}{3}\int pvn(p)dp$.The velocity information is also needed for obtaining pressure, therefore we divide our discussion into nonrelativistic and relativistic limit.

1.1 Nonrelativistic complete degeneracy

For a nonrelativistic particle, we can take $v = \frac{p}{m}$, then the pressrue is

$$P_e = \frac{8\pi}{15mh^3}p_0^5 = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n_e^{\frac{5}{3}} = 1.004\times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}} \quad dyne/cm^2$$

where the second equality replace p_0 by n_e, and the third equality represent n_e by mass density ρ. μ_e is the mean molecular weight per electron. Note that the pressure in nonrelativistic degenerated gas is proportional to the density to the power of $\frac{5}{3}$.

1.2 Relativisitic complete degeneracy

In the relativistic particle, the momentum is described by $p = \frac{m_0 v}{\left[1-\left(\frac{v}{c}\right)^2\right]^{\frac{1}{2}}}$, so $v = \frac{p/m_0}{\left[1+(p/m_0c)^2\right]^{\frac{1}{2}}}$. Plug the velocity in and evaluate th integral, we have

$$\begin{aligned} P_e &= \frac{\pi m^4c^5}{3h^3}f(x) \ \ , \ \ x = \frac{p_0}{mc} \\ f(x) &\approx 2x^4 - 2x^2 + ... \text{ as } x \to \infty \end{aligned}$$

Note that x can also be represented as a function of density if we replace p_0 by n_e, we now have $x = 1.009\times 10^{-2}\left(\frac{\rho}{\mu_e}\right)^{\frac{1}{3}}$. As x getting large, which corresponds to relativisitic velocity and high density, $f(x) \sim x^4$, and the pressure in relativistic degenerated gas is proportional to the density to the power of $\frac{4}{3}$.

Let's consider the validity of each case. As the density increase, the gas transform from ideal gas to nonrelativistic degenerated gas, and then to relativistic degenerated gas. The boundary between nondegenerated and degenerated gas can be obtained by this equality

$$\left(\frac{N_0\rho}{\mu_e}\right)kT = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\left(\frac{N_0\rho}{\mu_e}\right)^{\frac{5}{3}} \qquad \left(\frac{NkT}{V} = P_{e,\text{NR degenerated}}\right)$$

From numerical calculation, the degenerated pressure is larger then nondegenerated pressure when $\frac{\rho}{\mu_e} \succ 2.4 \times 10^{-8} T^{\frac{3}{2}}\ g/cm^3$. In the center of a white dwarf, $\frac{\rho}{\mu_e} \sim 10^6$ and $T \sim 10^6$, so this condition is satisfied. As we knew, the center of a white dwarf is indeed dominated by degeneracy pressure.

.On the other hand, we can find the density range for relativisic degenerated

gas as following. From the triangle relationship between total energy, rest mass energy and momentum, we have $pc = \frac{v}{c}E_{tot}$. In the relativistic limit, we can assume that $\frac{v}{c} \to 1$ and $pc \sim 2m_0c^2$, which means the total energy is much greater than the rest mass energy. The highest energy level of the degenerated gas should at least satisfy this condition. Therefore,

$$p_0c = hc\left(\frac{3}{8\pi}n_e\right)^{\frac{1}{3}} = 5.15 \times 10^{-3}\left(\frac{\rho}{\mu_e}\right)^{\frac{1}{3}}\ MeV$$

put in $p_0c \sim 2m_0c^2$ for m_0 equal to electron's rest mass, we finally have $\frac{\rho}{\mu_e} \succ 7.3 \times 10^6\ g/cm^3$ for relativistic degenerated electron gas. Comparing this high density with ideal gas, as we did in nonrelativistic part, the equality happpens when $T \sim 10^9\ K$. In astronomy, there are seldom cases with this high density and temperature, so we often ignore the relativistic discussion.

2 Partial degeneracy

In the real situation, the temperature is finite, and therefore not all the electrons are degenerated. Now I consider the partial degeneracy of electrons. As I discussed in the end of last part, we usually ignore the situation when the particles are relativistic, so we take $v = \frac{p}{m}$ as in the nonrelativistic complete degenerated gas. The energy of the electron particle can be evaluated as $\frac{p^2}{2m}$. Also, because the electrons are not complete degenerated, there is not strict upper limit of energy level. We have to integrate the density as well as the pressure over all energy levels by extending the momentum upper limit to infinity.

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2dp}{\exp\left(\alpha + p^2/2mkT\right) + 1}\ ,\ P_e = \frac{8\pi}{3h^3m}\int_0^\infty \frac{p^4dp}{\exp\left(\alpha + p^2/2mkT\right) + 1}$$

by defining $u = \frac{p^2}{2mkT}$ and seperate out the integral, we have

$$\begin{aligned} n_e &= \frac{4\pi}{h^3}\left(2mkT\right)^{\frac{3}{2}} F_{\frac{1}{2}}\left(\alpha\right)\ ,\ P_e = \frac{8\pi}{3h^3m}\left(2mkT\right)^{\frac{3}{2}} F_{\frac{3}{2}}\left(\alpha\right) \\ \text{where } F_{\frac{1}{2}}\left(\alpha\right) &= \int_0^\infty \frac{u^{\frac{1}{2}}du}{\exp\left(\alpha + u\right) + 1} \text{ and } F_{\frac{3}{2}}\left(\alpha\right) = \int_0^\infty \frac{u^{\frac{3}{2}}du}{\exp\left(\alpha + u\right) + 1} \end{aligned}$$

As we mentioned before, α is a parameter depends on density and temperature which approaches negative infinity in complete degenrated gas. Examing $F_{\frac{1}{2}}$ and $F_{\frac{3}{2}}$ at different α, we can measure the extent of degeneracy by combining P_e and n_e

$$P_e = n_e kT \left(\frac{2F_{\frac{3}{2}}}{3F_{\frac{1}{2}}} \right)$$

$\frac{2F_{\frac{3}{2}}}{3F_{\frac{1}{2}}}$ gives us the extent of the electron pressure differs from that of nondegenerated gas. From numerical calculation, the gas becomes degenerated after $\alpha \prec 2$.

When we consider the total pressure in a star, we should include both the contribution from electron degeneracy pressure and also from ordinary particles(which are nuclear gas). We have $P_{gas} = P_e + \frac{N_0 \rho}{\mu_i} kT$ where μ_i is the mean molecular weight of these nuclear gas(ions).

For example, I consider gas composed of C^{12} and density of $2.5 \times 10^5\ g/cm^3$ at $T = 10^8\ K$.The corresponding $\alpha \approx -12$ and $P_e \approx 10^{22}\ dyne/cm^2$. On the other hand, the pressure from ions is about $10^{21}\ dyne/cm^2$. In this case, the degeneracy pressure is ten times larger than the pressure from ions. In the high evolusion stages of stars such as white dwarf, the degeneracy is much higher(α is much more negative), and the ratio between degeneracy pressure and ion pressure is much larger. In conlusion, degeneracy pressure dominates the gas pressure in the high evolution satges of stars.

Although the temperature is finite, we can still consider the electrons in white dwarf or even higher evolution stages of stars as complete degeneracy. Consider a white dwarf with density $10^9\ g/cm^3$ at $T = 10^7\ K$, the corresponding $\alpha \approx -3 \times 10^5$. However, because it's still in nonrelativistic region, $\frac{E}{kT} \ll \alpha$ and the number density distribution of electrons is therefore much like that of complete degeneracy. If we assume complete degeneracy in the stars, the pressure becomes independent of temperature.

Part II

Pressure ionization

In previous part, we consider all particle as perfect gas, which means that there is not interaction between particles. However, in the interior of white dwarf, the density is so high that the particle seperation are compressed. The electron energy levels of nearby atoms would overlapped. Because electron is fermi gas, the mutual wave function of overlapping electrons must be antisymmetric. Due to Pauli exclusion principle, the many degenerated energy levels of discrete energy reform a continuous band of energy shared by all atoms, similar to that in a metal. The electrons are then said to be free or ionized, and indeed the wave function of electron can be expressed by that of free electron. This phenomenon is what we called pressure ionization.

Take a as the radius of an atom, pressure ionization should satisfy this relation

$$\frac{a^3}{\gamma_1}\frac{\rho}{Am_H} = 1$$

where A denotes the atomic weight of atom, m_H is the mass of hydrogen. γ_1 is a factor of the order of unity and is related mainly to the lattice arrangement of atoms. This relation means that the atom cells occupy all the spacial volume in the gas.

In general, some of these electrons would be bound and the other would be free. Both of them are degenerated electrons and can be assumed to be uniformly distributed. As I discussed in last part, the electron gas can also be viewed as complete degeneracy, so we can express the total kinetic energy of the gas as(from statistical mechanics)

$$T = \frac{3}{5}NE_f = \frac{3}{10}\frac{h^2}{m}\left(3\pi^2 n_e\right)^{\frac{2}{3}} N$$

where E_f is the fermi energy, n_e is the electron number density(both bound and free electrons), and N is the total number of electrons. The total number N of electron is $Z\left(\frac{\rho V}{Am_H}\right)$, where V is the total volume of the gas. The electron number density n_e can be estimated by $\frac{Z}{\gamma_2 a^3}$, where γ_2 is another factor of the order of unity related mainly to the arrangement of electrons. The radius of an atom can also be written as a funiction of density from the relation of pressure ionization. The total kinetic enegy is then

$$T = \frac{3}{10}\frac{h^2}{m}\left(3\pi^2\frac{\rho Z}{\gamma_1\gamma_2 Am_H}\right)^{\frac{2}{3}} Z\left(\frac{\rho V}{Am_H}\right)$$

On the other hand, the potential energy of the electrons is mainly from coulomb potential. We have to consider both the coulomb force from neucleus and surounding electrons

$$U = U_{ee}+U_{eZ} = (Ze)^2\int_0^a\left(\frac{r}{a}\right)^3\frac{4\pi r^2dr}{\frac{4}{3}\pi a^3} - (Ze)^2\int_0^a\frac{1}{r}\frac{4\pi r^2dr}{\frac{4}{3}\pi a^3} = \left(\frac{3}{5}-\frac{3}{2}\right)\frac{(Ze)^2}{a} = -\frac{9}{10}\frac{(Ze)^2}{a}$$

Substituting a from pressure ionization relation and mutiply the potential energy by the number of atoms, we finally have

$$U_{tot} = -\frac{9}{10}(Ze)^2\left(\frac{\gamma_1 Am_H}{\rho}\right)^{\frac{1}{3}}\frac{\rho V}{Am_H}$$

In oder to heading to virial theorem, we still need the pressure of electrons, which can be obtained by the discussion of nonrelativistic complete degeneracy. Notice that the pressure is supported only by free electrons, so we denote the number density of free electron as $n_{e,free}$, and the pressure is

$$P_e = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n_{e,free}^{\frac{5}{3}}$$

In this case it's proper to ignore the pressure from nuclear gas as the degeneracy is high and the atoms are highly compressed.

Before working on virial theorem, I have one thing to mention. If the atom is r times ionized, then there are r free electrons and $Z - r$ bound electrons. The number density of free electrons is thereby

$$n_{e,free} = r\frac{\rho}{Am_H} = \frac{\rho}{\mu_{e,free}m_H}$$

where is $\mu_{e,free}$ the mean molecular weight of free electron and is equal to $\frac{A}{r}$. $\mu_{e,free}$ provides us a way to measure the extent of ionization.

The virial theorem is $2T + U = 3PV$. After some calculations, we have

$$\mu_{e,free} = \frac{A/Z\left(\gamma_1\gamma_2\right)^{\frac{2}{3}}}{\left[1 - \frac{(\gamma_1\gamma_2)^{\frac{2}{3}}}{\gamma_1^{\frac{1}{3}}}\left(\frac{\Delta ZA}{\rho}\right)^{\frac{1}{3}}\right]^{\frac{3}{5}}}$$

where Δ is the collection of some constant. From here we have no information about γ_1 and γ_2, but we can roughly take them to be 1 as they are in the order of unity. Then

$$\mu_{e,free} = \frac{\mu_0}{\left[1 - \left(\frac{\Delta ZA}{\rho}\right)^{\frac{1}{3}}\right]^{\frac{3}{5}}}$$

where $\mu_0 = \frac{A}{Z}$ represents the mean molecular weight of free electron for fully ionized atom. The density ρ can also be replace by pressure P_e. As a result, the degree of ionization only depends on pressure or density. Thus we called it pressure ionization.

Now I discuss pressure ionization in another way. To ionize an electron from an atom, the kinetic energy from degeneracy should overcome the coulomb potential energy. To satisfy this condition, we need

$$|U| \approx T$$

which means

$$\frac{9}{10}\frac{(Ze)^2}{a} = \frac{3}{5}E_f \Longrightarrow \frac{9}{10}(Ze)^2\left(\frac{Am_H}{\rho}\right)^{\frac{1}{3}} = \frac{3}{10}\frac{h^2}{m}\left(3\pi^2\frac{\rho Z}{Am_H}\right)^{\frac{2}{3}}$$

so we can obtain ρ as a function of only A and Z, which means the interior density of white dwarf is almost determined by the composition elements.

Let's now apply our analysis to a real case. We can find the boundry of each states on $\rho - T$ plot as following:

Assume the composition ratio of hydrogen and helium are 0.75 and 0.25, repectively. The mean molecular weight per ions is then about 0.59, and the mean molecular weight per electron is 1.14.

1. The boundry between ideal gas and relativistic complete degeneracy: $P_{gas} = P_{e,R}$

$$\log T = 6.87 + \frac{1}{3}\log\rho$$

2. The boundry between ideal gas and nonrelativistic complete degeneracy: $P_{gas} = P_{e,NR}$

$$\log T = 4.75 + \frac{2}{3}\log\rho$$

3. The boundry between relativistic complete degeneracy and nonrelativistic complete degeneracy: $P_{e,R} = P_{e,NR}$

$$\log\rho = 6.34$$

4. The boundary of pressure ionization:$|U| \approx T$

$$\log\rho \succ 3$$

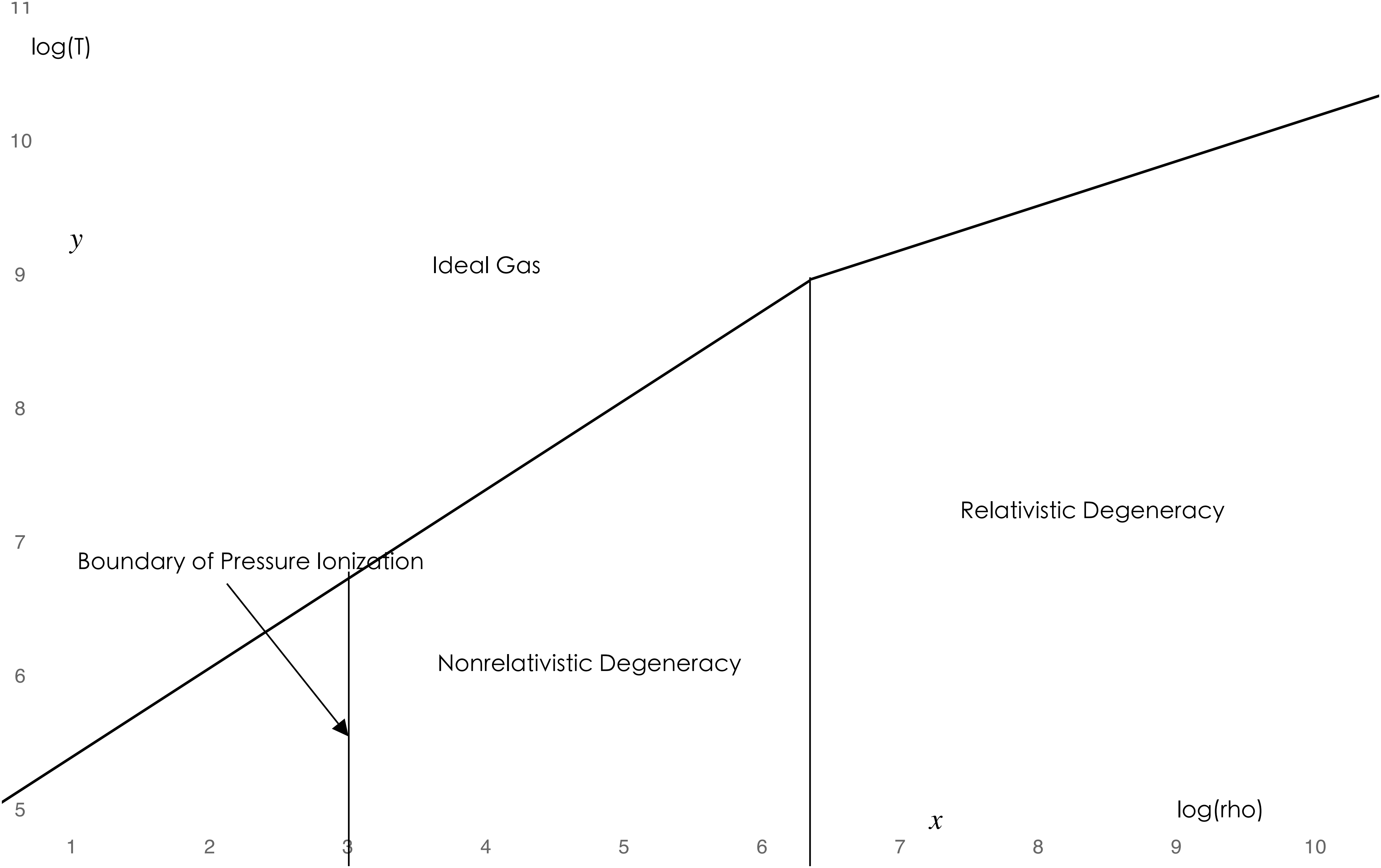
I made a $\rho - T$ plot showing the relationship between each states.

Conclusion 1 *The equation of state of degenerated gas plays an important role in Astronomy. Through my examination, I collect and discuss the applicable ranges of each kinds of degeneracy. The discussion bases on Fermi statistics and some straightforward assupmtions but works quite well in modern astronomy. There are still some interesting cases I didn't include in this project. For example, neutron as fermi gas also degenerated when the density and mass get much higher than white dwarf, and we call it neutron star. The physics in neutron degeneracy is still similar to that in electron degeneracy which we have discussed. Another example is the derivation of Chandrasekhar mass limit of white dwarf. The balance between gravitational collapse and degeneracy pressure gives an upper limit to total mass of white dwraf. The thermophysics of degenerated gas also has influence on the evolution of stars. In conclusion, discussion of the properties of degenerated fermi gas gives us a pretty good picture of physics in stars.*

References

[1]

1. Principle of Stellar Evolution and Nucleosynthesis/ Donald D. Clayton
2. The theory of pressure-ionization and its application/ D.S.Kothari
3. Statistical Mechanics/ R.K.Pathria
4. Modern Quantum Mechanics/ J.J.Sakurai



log(T)
11
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9
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y
Ideal Gas
Relativistic Degeneracy
Boundary of Pressure Ionization
Nonrelativistic Degeneracy
x
log(rho)
1
2
3
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