ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

PROBLEM SET #8 AND ANSWERS - DUE TUESDAY, DECEMBER 2

This problem set will be worth 5 points. Notation for problems: G = Goldstein *et al.*; PWJ: Porter W. Johnson's draft; FW = Fetter and Walecka. First number is chapter; second is problem number.

(1) (2 points) [G 9.39]: (a) Show from the Poisson bracket condition for conserved quantities that the Laplace-Runge-Lenz vector $\vec{A}$,

$$\vec{A} = \vec{p} \times \vec{L} - \frac{mk\vec{r}}{r} \ , \tag{1}$$

is a constant of the motion for the Kepler problem.

Answer: The time-dependence of a function $A(q_i, p_i)$ is given by its Poisson bracket with the Hamiltonian (summation implied):

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial A}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} = [A, H]_{q,p} \ . \tag{2}$$

The Laplace-Runge-Lenz vector (1) may be written

$$A_m = \epsilon_{mij} p_i \epsilon_{jln} r_l p_n - \frac{mkr_m}{r} = r_m p^2 - p_m \vec{r}\cdot\vec{p} - \frac{mkr_m}{r} \ . \tag{3}$$

For a central $1/r$ potential, with $H = p^2/(2m) - k/r$, one has ($q_i = r_i$)

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m} \ , \quad \frac{\partial H}{\partial q_i} = \frac{kr_i}{r^3} \ , \quad \text{so} \tag{4}$$

$$\begin{aligned} [A_m, H] &= [\delta_{im} p^2 - p_m p_i - \frac{mk}{r}(\delta_{im} - \frac{r_i r_m}{r^2})]\frac{p_i}{m} \\ &- [2r_m p_i - \delta_{im}\vec{r}\cdot\vec{p} - p_m r_i]\frac{kr_i}{r^3} \\ &= k\frac{\vec{r}\cdot\vec{p}r_m}{r^3}(1 - 2 + 1) = 0 \ . \end{aligned} \tag{5}$$

(b) Verify the Poisson bracket relations for the components of $\vec{A}$ as given by Goldstein *et al.*'s Eq. (9.131):

$$[A_i, L_j]_{q,p} = \epsilon_{ijk} A_k \ . \tag{6}$$

Answer: We need to calculate

$$[A_i, L_j]_{q,p} = \frac{\partial A_i}{\partial q_m}\frac{\partial L_j}{\partial p_m} - \frac{\partial A_i}{\partial p_m}\frac{\partial L_j}{\partial q_m} \quad \text{where} \tag{7}$$

$$L_j = \epsilon_{jab} r_a p_b \quad , \quad \frac{\partial L_j}{\partial p_m} = \epsilon_{jam} r_a \quad , \quad \frac{\partial L_j}{\partial q_m} = \epsilon_{jmb} p_b \ . \tag{8}$$

We already showed that

$$\vec{p} \times \vec{L} = \vec{r} p^2 - \vec{p}(\vec{r}\cdot\vec{p}) \quad , \quad \frac{\partial}{\partial q_m}\left(\frac{r_i}{r}\right) = \frac{\delta_{mi}}{r} - \frac{r_m r_i}{r^3} \quad . \quad \text{Then} \tag{9}$$

$$\begin{aligned}[A_i, L_j]_{q,p} &= \left\{ \frac{\partial}{\partial q_m}[r_i p^2 - p_i(\vec{r}\cdot\vec{p})] - mk\left(\frac{\delta_{mi}}{r} - \frac{r_m r_i}{r^3}\right) \right\} \epsilon_{jam} r_a \\ &- \left\{ \frac{\partial}{\partial p_m}[r_i p^2 - p_i(\vec{r}\cdot\vec{p})] \right\} \epsilon_{jmb} p_b \end{aligned}$$

$$\begin{aligned} &= \left[\delta_{mi} p^2 - p_i p_m - mk\left(\frac{\delta_{mi}}{r} - \frac{r_m r_i}{r^3}\right)\right] \epsilon_{jam} r_a - [2p_m r_i - \delta_{im}(\vec{r}\cdot\vec{p}) - p_i r_m]\epsilon_{jmb} p_b \\ &= \epsilon_{jai} r_a p^2 - p_i \epsilon_{jam} r_a p_m - mk\epsilon_{jai}\frac{r_a}{r} + \epsilon_{jib} p_b(\vec{r}\cdot\vec{p}) + p_i \epsilon_{jmb} r_m p_b \\ &= \epsilon_{ija}\left[r_a p^2 - p_a(\vec{r}\cdot\vec{p}) - mk\frac{r_a}{r}\right] = \epsilon_{ija} A_a \ . \end{aligned} \tag{10}$$

In the next two problems, compare your answer with a result which can be obtained by elementary means.

(2) (2 points) [G 10.12]: A particle of mass m moves in a plane in a square well potential: $V(r) = -V_0$ for $0 < r < r_0$ and $V(r) = 0$ for $r > r_0$.

(a) Under what initial conditions can the method of action-angle variables be applied?

Answer: Action-angle variables can be applied here only for E between $-V_0$ and 0, in which case the motion is bounded. (See also below.)

(b) Assuming these conditions hold, use the method of action-angle variables to find the frequencies of the motion. (Find the period between the times at which the particle is at $r = r_0$.)

Answer: It is most convenient to write the Hamiltonian in terms of polar coordinates (r, ψ):

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) + V(r) \ . \tag{11}$$

The abbreviated action $S_0 = S + Et$ satisfies the Hamilton-Jacobi equation

$$\frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_0}{\partial \psi}\right)^2\right] + V(r) = E \ . \tag{12}$$

Separation of variables allows one to write $S_0 = S_r(r) + S_\psi(\psi)$, with $S_\psi(\psi) = \alpha_\psi \psi$, so

$$\frac{1}{2m}\left[S_r' + \frac{\alpha_\psi^2}{r^2}\right] + V(r) = E \quad , \tag{13}$$

where $S_r' = dS_r(r)/dr$. Thus

$$\frac{dS_r}{dr} = \sqrt{2m[E - V(r)] - \frac{\alpha_\psi^2}{r^2}} \quad . \tag{14}$$

This does not exist for any r unless the quantity under the square root sign can be positive for some r in $0 \le r \le r_0$. Define $2m[E + V_0] = \kappa^2$. Then

$$S_r(r) = \int_{r_{\min}}^{r} dr' \sqrt{\kappa^2 - \alpha_\psi^2/r^2} \ , \tag{15}$$

where $r_{\min} = \alpha_\psi/\kappa$. Thus the condition allowing the application of action-angle variables is $0 \le r_{\min} \le r_0$, or

$$0 \le \frac{\alpha_\psi}{\sqrt{2m(E + V_0)}} \le r_0 \ ; \quad -V_0 \le E_0 \le 0 \ . \tag{16}$$

The time elapsed for motion between r_0 and r is

$$t = \frac{\partial S_0}{\partial E} = \frac{\partial S_r}{\partial E} = \int_{r_{\min}}^{r} \frac{m dr'}{\sqrt{\kappa^2 - \alpha_\psi^2/r'^2}} \ . \tag{17}$$

The period between impacts on the wall is

$$\tau = 2\int_{r_{\min}}^{r_0} \frac{m r' dr'}{\sqrt{\kappa^2 r'^2 - \alpha_\psi^2}} = \frac{2m}{\kappa^2}\sqrt{\kappa^2 r_0^2 - \alpha_\psi^2} = \frac{1}{E + V_0}\sqrt{\kappa^2 r_0^2 - \alpha_\psi^2} \ . \tag{18}$$

This result also can be obtained directly using a free particle estimate. The particle's momentum is $\kappa = \sqrt{2m(E + V_0)}$. The angular momentum with respect to the origin is $\alpha_\psi = r_{\min}\kappa$. A particle traveling in a straight line between two points on the wall with minimum distance $r_{\min}$ from the origin must travel a distance $2d = 2\sqrt{r_0^2 - r_{\min}^2}$, by the Pythagorean Theorem, so its travel time is

$$\tau = \frac{2dm}{\kappa} = \frac{2m}{\kappa}\sqrt{r_0^2 - r_{\min}^2} = \frac{2m}{\kappa^2}\sqrt{\kappa^2 r_0^2 - \alpha_\psi^2} \ . \tag{19}$$

(3) (1 point) [G 10.13]: A particle moves in periodic motion in one dimension under the influence of a potential $V(x) = F|x|$, where F is a constant. Using action-angle variables, find the period of the motion as a function of the particle's energy.

Answer: Using the same method as above, for one-dimensional motion, one finds that a full period for motion is given by

$$\tau = 4\int_0^{x_{\max}} \frac{m dx}{\sqrt{2m(E - Fx)}} \ . \tag{20}$$

Change variables to $p^2 = 2m(E - Fx)$ and define $p_0^2 = 2mE$. Then

$$\tau = \frac{4}{F}\int_0^{p_0} dp = \frac{4p_0}{F} \ . \tag{21}$$

A direct calculation based on the constant acceleration F/m experienced by the particle between 0 and $x_{\max}$ gives $x_{\max} = (1/2)(F/m)(\tau/4)^2$. But $2mE = 2mFr_{\max}$, so $r_{\max} = E/F = p_0^2/(2mF)$, and

$$\left(\frac{\tau}{4}\right)^2 = \frac{2mr_{\max}}{F} = \left(\frac{p_0}{F}\right)^2 \quad \text{or} \quad \tau = \frac{4p_0}{F} \ . \tag{22}$$