ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

PROBLEM SET #7 AND ANSWERS - DUE TUESDAY, NOVEMBER 25

This problem set will be worth 6 points. Notation for problems: G = Goldstein *et al.*; PWJ: Porter W. Johnson's draft; FW = Fetter and Walecka. First number is chapter; second is problem number.

(1) (2 points) [G 8.20]: Obtain Hamilton's equations of motion for a plane pendulum of length l with mass point m whose radius of suspension rotates uniformly on the circumference of a vertical circle of radius a. Describe physically the nature of the canonical momentum and the Hamiltonian.

Answer: Let ϕ denote the angle that the point of suspension makes with the vertical direction, so that $\dot{\phi} = \omega$ is constant. The independent coordinates may be taken to be ϕ and the angle θ which the pendulum makes with respect to the vertical direction. Then, taking the "up" and "left" directions to be the x and y axes, respectively, the position of the pendulum at time t is

$$x(t) = a\cos\omega t - l\cos\theta \quad , \quad y(t) = a\sin\omega t - l\sin\theta \quad , \tag{1}$$

so that the kinetic energy $T = m(\dot{x}^2 + \dot{y}^2)/2$ is

$$\begin{aligned} T &= \frac{1}{2}m[(-\omega a\sin\omega t + l\sin\theta\dot{\theta})^2 + (\omega a\cos\omega t - l\cos\theta\dot{\theta})^2] \quad (2)\\ &= \frac{1}{2}m(\omega a)^2 + \frac{1}{2}ml^2\dot{\theta}^2 - m\omega al\dot{\theta}\cos(\theta - \omega t) \quad , \quad (3)\end{aligned}$$

while the potential energy is $V = mgx = mg(a\cos\omega t - l\cos\theta)$. Note the explicit time-dependence in T and V. Energy is being fed in and out of the pendulum by the driving circular motion. The canonical momentum corresponding to the coordinate θ is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} - m\omega al\cos(\theta - \omega t) \ . \tag{4}$$

The additional term vanishes when $\theta = \omega t \pm (\pi/2)$, i.e., when the pendulum is parallel to the driving point's velocity. The driving term then is not acting to change θ. The Hamiltonian is

$$\begin{aligned} H &= \dot{\theta}p_\theta - L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}m(\omega a)^2 + mg(a\cos\omega t - l\cos\theta) \quad (5)\\ &= \frac{1}{2ml^2}[p_\theta + m\omega al\cos(\theta - \omega t)]^2 - \frac{1}{2}m(\omega a)^2 + mg(a\cos\omega t - l\cos\theta) \ . \quad (6)\end{aligned}$$

Note that there is no term linear in $\dot{\theta}$, in contrast to the Lagrangian $L = T - V$. Hamilton's equations are

$$\begin{aligned}
\frac{\partial H}{\partial p_\theta} &= \frac{1}{ml^2}[p_\theta + m\omega al\cos(\theta - \omega t)] = \frac{1}{ml^2}(ml^2\dot{\theta}) = \dot{\theta} \ , \qquad (7)\\
\frac{\partial H}{\partial \theta} &= \frac{\omega a}{l}\sin(\theta - \omega t)[p_\theta + m\omega al\cos(\theta - \omega t)] + mgl\sin\theta = -\dot{p}_\theta \ . \qquad (8)
\end{aligned}$$

The first term in the second equation is due to the coupling of the pendulum to the external system. This is similar to the way in which the electromagnetic interaction is introduced in classical electrodynamics, quantum mechanics, or quantum field theory.

(2) (2 points) [G 8.24]: A uniform cylinder of radius a and density ρ is mounted so as to rotate freely around a vertical axis. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction. Suppose a particle starts at rest on the top of the cylinder and slides down under the influence of gravity. Using any set of coordinates, arrive at a Hamiltonian for the combined system of particle and cylinder, and solve for the motion of the system.

Answer: Let h be the height of the cylinder and ρ be its mass density. Its moment of inertia is $I_c = (\pi a^4 h\rho)/2$. Call ψ the particle's angular coordinate with respect to the point at the top of the cylinder where it starts out. Defining z as the vertical direction with $z = 0$ corresponding to the top of the cylinder, one then has $z = -\alpha\psi$, where α is some constant. Let θ denote the angle of rotation of the cylinder in an inertial frame. Then the angular coordinate ϕ of the mass point m in this inertial frame is $\phi = \psi + \theta$. The kinetic energies of the cylinder and the mass point are

$$T_{\text{cyl}} = \frac{1}{2}I_c\dot{\theta}^2 \ , \quad T_m = \frac{1}{2}m(\dot{z}^2 + a^2\dot{\phi}^2) \ . \qquad (9)$$

Thus the Lagrangian may be written

$$L = \frac{1}{2}I_c\dot{\theta}^2 + \frac{1}{2}m[\alpha^2\dot{\psi}^2 + a^2(\dot{\psi} + \dot{\theta})^2] + mg\alpha\psi \ . \qquad (10)$$

The Lagrangian does not contain θ, so $p_\theta = \partial L/\partial\dot{\theta} = \text{const.}$:

$$p_\theta = I_c\dot{\theta} + ma^2(\dot{\psi} + \dot{\theta}) \quad \Rightarrow \quad \dot{\theta} + \frac{ma^2}{I_c + ma^2}\dot{\psi} = \quad \text{const.} \qquad (11)$$

Since $\dot{\theta}(0) = \dot{\psi}(0) = 0$, $p_\theta = 0$, so $\dot{\theta} = -[ma^2/(I_c + ma^2)]\dot{\psi}$. Furthermore

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = m\alpha^2\dot{\psi} + ma^2(\dot{\psi} + \dot{\theta}) = \dot{\psi}\left[m\alpha^2 + \frac{ma^2 \ I_c}{I_c + ma^2}\right] \ . \qquad (12)$$

The only ψ dependence in $H \equiv \dot{\theta} p_\theta + \dot{\psi} p_\psi - L$ comes from $V = mgz = -mg\alpha\psi$, so

$$\dot{p}_\psi = -\frac{\partial H}{\partial \psi} = mg\alpha \quad \Rightarrow \quad \ddot{\psi}\left[m\alpha^2 + \frac{m\alpha^2\ I_c}{I_c + ma^2}\right] = mg\alpha \ , \tag{13}$$

implying a uniform acceleration. Integrating once, we find

$$\dot{\psi} = \frac{g\alpha(I_c + ma^2)t}{(\alpha^2 + a^2)I_c + ma^2\alpha^2} \ ; \ \dot{\theta} = -\frac{ma^2 g\alpha t}{(\alpha^2 + a^2)I_c + ma^2\alpha^2} \ , \tag{14}$$

and a second integration is straightforward.

(3) (2 points) [FW 6.6]: Construct from first principles the Hamiltonian for a one-dimensional harmonic oscillator of mass m and spring constant k. Determine the value of the constant C such that the following equations define a canonical transformation from the old variables (q, p) to the new vaeiables (Q, P):

$$Q = C(p + im\omega q) \qquad \text{and} \qquad P = C(p - im\omega q) \ , \tag{15}$$

where $\omega = (k/m)^{1/2}$. What is the generating function $S(q, P)$ for this transformation? Find Hamilton's equations of motion for the new variables and integrate them. Hence find the solution to the original problem. (This transformation defines the creation and annihilation operators of quantum theory.)

Answer: In terms of the old variables, we have

$$T = \frac{1}{2}m\dot{q}^2 \ , \ \ V = \frac{1}{2}kq^2 \ , \ \ L = T - V \ , \ \ p = \frac{\partial L}{\partial q} = m\dot{q} \ , \tag{16}$$

$$H = p\dot{q} - L = \frac{p^2}{2m} + \frac{k}{2}q^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \ . \tag{17}$$

A canonical transformation should preserve Poisson brackets: $[P, Q]_{p,q} = [p, q]_{p,q} = 1$:

$$\begin{aligned} [P, Q]_{p,q} &= C^2\left[\frac{\partial}{\partial p}(p - im\omega q)\frac{\partial}{\partial q}(p + im\omega q) - \frac{\partial}{\partial q}(p - im\omega q)\frac{\partial}{\partial p}(p + im\omega q)\right] \ . \\ &= C^2(2im\omega) \quad \Rightarrow \quad C^2 = (2im\omega)^{-1} \ . \end{aligned} \tag{18}$$

A generating function of the form $S(q, P)$ should satisfy (see, e.g., Table 9.1 on p. 373 of Goldstein *et al.*)

$$p = \frac{\partial S(q, P)}{\partial q} \quad , \quad Q = \frac{\partial S(q, P)}{\partial P} \ . \tag{19}$$

Since the canonical transformation involves linear functions, it is natural to try a homogeneous quadratic form for the generating function: $S(q, P) = aq^2 + bqP + cP^2$. Then

$$p = \frac{\partial S}{\partial q} = 2aq + bP = 2aq + bC(p - im\omega q) \ . \tag{20}$$

This implies $1 = bC$, $0 = 2a - bCim\omega = 2a - im\omega$. Thus $a = im\omega/2$, $b = 1/C = \sqrt{2im\omega}$ (choosing the positive square root). Similarly

$$Q = \frac{\partial S}{\partial P} = bq + 2cP \quad \Rightarrow \quad C(p + im\omega q) = bq + 2cC(p - im\omega q) \ , \tag{21}$$

and comparing the coefficients of p on left and right, we find $c = 1/2$, which can be shown to giving matching coefficients of q as well. Thus

$$S(q, P) = \frac{im\omega}{2} q^2 + \sqrt{2im\omega} qP + \frac{1}{2} P^2 \ . \tag{22}$$

The new Hamiltonian $H'(P, Q)$ is just $H(p, q)$, since the canonical transformation is time-independent:

$$H'(P, Q) = H(p, q) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = \frac{1}{2m} \left[\frac{Q + P}{2C} \right]^2 + \frac{1}{2} m\omega^2 \left[\frac{Q - P}{2im\omega C} \right]^2 = i\omega QP \ . \tag{23}$$

The new Hamilton's equations are

$$\dot{P} \ = \ -\frac{\partial H'}{\partial Q} = -i\omega P \quad \Rightarrow \quad P(t) = P(0) e^{-i\omega t} \tag{24}$$

$$\dot{Q} \ = \ \frac{\partial H'}{\partial P} = i\omega Q \quad \Rightarrow \quad Q(t) = Q(0) e^{i\omega t} \ . \tag{25}$$

Then $q(t) = [Q(0)e^{i\omega t} - P(0)e^{-i\omega t}]/(2Cim\omega)$, where $Q(0) = C[p(0) + im\omega q(0)]$ and $P(0) = C[p(0) - im\omega q(0)]$. This then yields the familiar solution

$$q(t) = q(0) \cos \omega t + \frac{p(0)}{m\omega} \sin \omega t \ . \tag{26}$$