ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

PROBLEM SET #4 AND ANSWERS - DUE TUESDAY, NOVEMBER 4

This problem set will be worth 8 points. Notation for problems: G = Goldstein *et al.*; PWJ: Porter W. Johnson's draft; FW = Fetter and Walecka. First number is chapter; second is problem number.

(1) (3 points) [G 6.3]: A bead of mass $m$ is constrained to move on a hoop of radius $R$. The hoop rotates with constant angular velocity $\omega$ around a diameter of the hoop, which is a vertical axis (line along which gravity acts).

**(a)** Set up the Lagrangian and obtain the equations of motion of the bead.

**Answer**: Let the position of the bead on the hoop be described by an angle $\theta$, where $\theta = 0$ corresponds to the bottom of the hoop. Then the kinetic energy is $T = (m/2)[R^2\dot{\theta}^2 + (R\omega\sin\theta)^2]$, while the potential energy is $V = -mgR\cos\theta$. Thus the Lagrangian is

$$L = \frac{m}{2}[R^2\dot{\theta}^2 + (R\omega\sin\theta)^2] + mgR\cos\theta \ . \tag{1}$$

Total energy $E = T + V$ is not conserved because energy is transferred to the bead by the rotating hoop. However, the $m(R\omega\sin\theta)^2/2$ term in $L$ may be regarded as a contribution to an effective potential $V_{\rm eff}$. This is equivalent to the conservation of the quantity $\dot{\theta}(\partial L/\partial\dot{\theta}) - L$. It is most straightforward to simply write the equation of motion for $\theta$ in terms of Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mR^2\ddot{\theta} = \frac{\partial L}{\partial\theta} = -\frac{\partial V_{\rm eff}}{\partial\theta} \ , \tag{2}$$

where

$$V_{\rm eff} \equiv -\frac{1}{2}mR^2\sin^2\theta\omega^2 - mgR\cos\theta \ \Rightarrow \ \frac{\partial L}{\partial\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \ . \tag{3}$$

**(b)** Find the critical angular velocity $\Omega$ below which the bottom of the hoop provides a stable equilibrium for the bead.

**Answer**: Expanding the equation of motion about $\theta = 0$, we find $\ddot{\theta} \simeq \theta[\omega^2 - (g/R)]$, so the critical angular velocity below which the equilibrium is stable at $\theta = 0$ is $\omega = \Omega = \sqrt{g/R}$.

**(c)** Find the stable equilibrium position for $\omega > \Omega$.

**Answer**: Stable equilibrium points are those for which $\partial V_{\rm eff}/\partial\theta = 0$, $\partial^2 V_{\rm eff}/\partial\theta^2 > 0$. We have $\partial V_{\rm eff}/\partial\theta = -mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta$. One root occurs at $\sin\theta = 0$; the above expansion about $\theta = 0$ shows that this corresponds to stable equilibrium only for $\omega < \Omega$. Another root occurs for $-R\omega^2\cos\theta + g = 0$, which has a solution for $\cos\theta = (g/R\omega^2) = (\Omega/\omega)^2$ as long as $\omega > \Omega$.

(2) (3 points) [G 6.10]: **(a)** Three equal mass points have equilibrium positions at the vertices of an equilateral triangle. They are connected by equal springs that lie along the arcs of the circle circumscribing the triangle. Mass points and springs are constrained to move only on the circle so that, for example, the potential energy of a spring is determined by the arc length covered. Determine the eigenfrequencies and normal modes of small oscillations in the plane. Identify physically any zero frequencies.

**Answer**: Let the angular coordinates of the mass points on the circle of radius $R$ be denoted by $\theta_{1,2,3}$. Then the kinetic energy is $T = (m/2)(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + \dot{\theta}_3^{\,2})$, while the potential energy is $V = (k/2)[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 + (\theta_3 - \theta_1)^2]$, where $k$ is the spring constant. Setting up the equations of motion, letting each $\theta_i$ have a time dependence $\sim e^{-i\omega t}$, and defining the matrix

$$K \equiv k \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} , \tag{4}$$

one must solve the eigenvalue problem $\mathrm{Det}(K - \omega^2 m\mathcal{I}) = 0$, where $\mathcal{I}$ is the $3 \times 3$ unit matrix. Defining $\lambda \equiv \omega^2 m/k$, one finds that this is equivalent to $\lambda(\lambda - 3)^2 = 0$, which has a single root at $\lambda = 0$ and a double root at $\lambda = 3$. The root at $\lambda = 0$ corresponds to the eigenvector $X_1^T = [1\ 1\ 1]$, i.e., a uniform rotation of all the masses. The other two modes are degenerate and may be taken as any two mutually orthogonal linear combinations of two orthonormal basis vectors orthogonal to $X_1$, e.g., $X_2^T = [1\ -1\ 0]$ and $X_3^T = [2\ -1\ -1]$.

**(b)** Suppose one of the springs has a change in force constant $\delta k$, the others remaining unchanged. To first order in $\delta k$, what are the changes in the eigenfrequencies and normal modes?

**Answer**: Define $\eta \equiv \delta k/k$. Letting the force constant between masses 1 and 2 be changed to $k + \delta k$, one now has

$$K = k \begin{bmatrix} 2+\eta & -(1+\eta) & -1 \\ -(1+\eta) & 2+\eta & -1 \\ -1 & -1 & 2 \end{bmatrix} . \tag{5}$$

The determinant of this is still zero so one eigenvector remains $X_1^T = [1\ 1\ 1]$ with eigenvalue $m\omega^2 = 0$. The characteristic equation for $\lambda = \omega^2 m/k$ becomes $\lambda[\lambda^2 - (6 + 2\eta)\lambda + (9 + 6\eta)] = 0$. Let the non-zero roots be expressed as $\lambda = 3 + \epsilon$; one then finds $\epsilon^2 - 2\epsilon\eta = 0$, whose roots are $\epsilon = 0$ and $\epsilon = 2\eta$. The eigenvalue $\lambda = 3$ ($\epsilon = 0$) is unchanged by the perturbation. It corresponds to the eigenvector $X^T = [1\ 1\ -2]$ where masses 1 and 2 move together, so the oscillation is insensitive to the perturbation. The eigenvalue $\lambda = 3 + 2\eta$ corresponds to the orthogonal mode $X^T = [1\ -1\ 0]$ in which masses 1 and 2 move opposite to one another while mass 3 does not move.

**(c)** Suppose what is changed in the mass of one of the particles by an amount $\delta m$. Now how do the normal eigenfrequencies and normal modes change?

**Answer**: Let particle 3 be the one whose mass is changed and define $\eta \equiv \delta m/m$. The characteristic equation which now must be solved is

$$\text{Det}\begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda(1+\eta) \end{bmatrix} = 0 \ , \quad \text{or} \ \ \lambda[\lambda^2(1+\eta) - (6+4\eta)\lambda + (9+3\eta)] = 0 \ . \tag{6}$$

This continues to have the solution $\lambda = 0$ corresponding to uniform motion of all the masses, $X^T = [1\ 1\ 1]$, The other roots turn out to be $\lambda = (3 + 2\eta \pm \eta)/(1 + \eta)$. The root $\lambda = 3$ corresponds to the eigenvector $X^T = [1\ -1\ 0]$ in which particle 3 does not move, so changing its mass has no effect on $\lambda$. The root $\lambda = (3+\eta)/(1+\eta) \simeq 3-2\eta$ corresponds to the eigenvector $X^T = [1\ 1\ -2]$. The frequency goes down when particle 3 is heavier.

(3) (2 points) [FW 4.8]: Four massless rods of length $L$ are hinged together at their ends to form a rhombus. A particle of mass $M$ is attached at each joint. The opposite corners of the rhombus are joined by springs, each with a spring constant $k$. In the equilibrium (square) configuration, the springs are unstretched. The motion is confined to a plane, and the particles move only along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about the equilibrium configuration.

**Answer**: Let the particles be labeled 1, 2, 3, 4, corresponding to the compass points East, North, West, and South. The symmetry of the problem dictates that 1 and 3 will oscillate only in the $x$ (East-West) direction, while 2 and 4 will oscillate only in the $y$ (North-South) direction. (We are not asked to consider other motions, such as overall rotations.) By the symmetry we have $x_3 = -x_1$ and $y_4 = -y_2$. Deviations from equilibrium positions are $x_1 = L/\sqrt{2} + \eta_1$, $y_2 = L/\sqrt{2} + \zeta_2$, so that

$$V = \frac{1}{2}k(\eta_1 - \eta_3)^2 + \frac{1}{2}k(\zeta_2 - \zeta_4)^2 = 2k(\eta_1^2 + \zeta_2^2) \ . \tag{7}$$

Moreover, the distance between each adjacent mass point is always constrained to be $L$, so

$$L^2 = \left(\frac{L}{\sqrt{2}} + \eta_1\right)^2 + \left(\frac{L}{\sqrt{2}} + \zeta_2\right)^2 \quad \text{or} \quad \zeta_2 = -\eta_1 \ . \tag{8}$$

This implies that $V = 4k\eta_1^2$. Since $T = (m/2)(\dot{\eta_1}^2 + \dot{\zeta_2}^2 + \dot{\eta_3}^2 + \dot{\zeta_4}^2) = 2m\dot{\eta_1}^2$, we have $2(2m\ddot{\eta_1}) = -2(4k\eta_1)$, or harmonic motion with angular frequency $\omega = \sqrt{2k/m}$.