ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

PROBLEM SET #3 - DUE TUESDAY, OCTOBER 28

This problem set will be worth 10 points. Notation for problems: G = Goldstein *et al.*; PWJ: Porter W. Johnson's draft; FW = Fetter and Walecka. First number is chapter; second is problem number.

(1) (3 points) [G 3.2]: By expanding $e\sin\psi$ in a Fourier series in ωt, show that Kepler's equation has the formal solution

$$\psi = \omega t + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne)\ \sin n\omega t, \tag{1}$$

where J_n is the Bessel function of order n. For small argument, the Bessel function can be approximated in a power series of the argument. Accordingly, from this result derive the first few terms in the expansion of ψ in orders of e. Note: give your result up to and including order e^3. It is helpful here to use Bessel's integral,

$$J_n(ne) = \frac{1}{\pi}\int_0^{\pi} \cos[n(\psi - e\sin\psi)]\ d\psi\ . \tag{2}$$

Answer: Kepler's equation is

$$\omega t = \psi - e\sin\psi \equiv M\ . \tag{3}$$

As $\sin\psi$ and ψ are odd in ψ, one can expand $e\sin\psi$ in a Fourier sine series:

$$e\sin\psi = \sum_{n=1}^{\infty} a_n \sin(n\omega t)\ . \tag{4}$$

Multiply this equation by $\sin kM$ and integrate with respect to M from 0 to 2π:

$$e\int_0^{2\pi} \sin\psi\ \sin kM\ dM = \pi a_k\ ; \tag{5}$$

$$a_k = \frac{e}{\pi}\int_0^{2\pi} (1 - e\cos\psi) d\psi \sin k(\psi - e\sin\psi)\sin\psi\ . \tag{6}$$

Integrate by parts using $u\,dv = d(uv) - v\,du$ with $u = \sin\psi$, $dv = (1 - e\cos\psi)d\psi\ \sin k(\psi - e\sin\psi)$, so $v = -(1/k)\cos k(\psi - e\sin\psi)$. The term $\int d(uv)$ vanishes because $uv = 0$ at 0 and 2π, so we have

$$a_k = \frac{e}{\pi k}\int_0^{2\pi} d\psi\ \cos\psi\cos[k(\psi - e\sin\psi)]\ . \tag{7}$$

Now

$$\int_0^{2\pi} d\psi (1 - e\cos\psi)\cos[k(\psi - e\sin\psi)] = \frac{1}{k}\sin k(\psi - e\sin\psi)|_0^{2\pi} = 0\ . \tag{8}$$

Then, using the Bessel integral (2),

$$a_k = \frac{1}{\pi k}\int_0^{2\pi} d\psi \cos[k(\psi - e\sin\psi) = \frac{2}{k}J_k(ke) \ . \tag{9}$$

In order to expand ψ in powers of e, we need to expand the first few Bessel functions $J_n(x)$ up to third order in x:

$$J_1(x) \simeq \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^3 \ , \quad J_2(x) \simeq \frac{1}{2}\left(\frac{x}{2}\right)^2 \ , \quad J_3(x) \simeq \frac{1}{6}\left(\frac{x}{2}\right)^3 \ . \tag{10}$$

One then finds, collecting terms,

$$\psi = \omega t + e\left(1 - \frac{e^2}{8}\right)\sin\omega t + \frac{e^2}{2}\sin 2\omega t + \frac{3e^3}{8}\sin 3\omega t + \ldots \ . \tag{11}$$

(2) (3 points) [G 3.15]: A meteor is observed to strike Earth with a speed v, making an angle ϕ with the zenith. Suppose that far from Earth the meteor's speed was v' and it was proceeding in a direction making a zenith angle ϕ', the effect of Earth's gravity being to pull it into a hyperbolic orbit intersecting Earth's surface. Show how v' and ϕ' can be determined from v and ϕ in terms of known constants.

Answer: Let the gravitational potential be $V(r) = -\lambda/r = -(GMm/r)$, where m is the mass of the meteorite and M is the mass of the Earth. The orbit of the meteor is a hyperbola:

$$\frac{p}{r} = 1 + e\cos\theta \ , \quad p \equiv \frac{\ell^2}{m\lambda} \ . \tag{12}$$

where $e > 1$, and the asymptotes of the hyperbola occur for $\cos\theta = -1/e = \cos\phi'$, so that $\sec\phi' = -e$. The energy E and eccentricity e are, respectively,

$$E = \frac{1}{2}mv'^2 \ , \quad e = \sqrt{1 + \frac{2E\ell^2}{m\lambda^2}} \ . \tag{13}$$

Then $e^2 - 1 = (2E\ell^2)/(m\lambda^2) = (\ell v'/\lambda)^2$ and hence

$$\ell = \frac{\lambda}{v'}\sqrt{e^2 - 1} = \frac{\lambda}{v'}\tan\phi' \ . \tag{14}$$

At the radius R of the Earth, the kinetic energy of the meteorite is $(1/2)mv^2$, so

$$\frac{1}{2}mv'^2 = \frac{1}{2}mv^2 - \frac{\lambda}{R}, \tag{15}$$

permitting one to solve for v' in terms of v:

$$v' = \left(v^2 - \frac{2\lambda}{mR}\right)^{1/2} = \left(v^2 - \frac{2GM}{R}\right)^{1/2} \ . \tag{16}$$

The orbital angular momentum ℓ is given in terms of the zenith angle ϕ at impact by $\ell = mvR\sin\phi$. Knowing v' and ℓ, we use Eq. (14) to calculate ϕ'.

(3) (2 points) [FW 1.16]: A uniform beam of particles with energy E is scattered by a repulsive central potential $V(r) = \gamma/r^2$. Derive the differential elastic cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}} = \frac{\gamma\pi^2}{E\sin\theta}\frac{\pi-\theta}{\theta^2(2\pi-\theta)^2} \ . \tag{17}$$

Sketch carefully the angular dependence. Discuss the total cross section. What happens if the potential is attractive, that is, $\gamma < 0$?

Answer: The scattering angle θ as a function of impact parameter b is

$$\theta(b) = \pi - 2b\int_0^{u_0}\frac{du}{\sqrt{1-(\gamma/E)u^2-b^2u^2}} \ ; \ \ u_0 \equiv [(\gamma/E)+b^2]^{-1/2} \ . \tag{18}$$

Defining $(\gamma/E)+b^2 = \beta^2$ and $\sin\psi = \beta u$, the integral is

$$\theta(b) = \pi\left(1-\frac{b}{\beta}\right) = \pi\left(1-\frac{b}{\sqrt{b^2+(\gamma/E)}}\right) \ . \quad \text{Then} \tag{19}$$

$$\frac{d\theta}{db} = \frac{-\pi\gamma/E}{[b^2+(\gamma/E)]^{3/2}} \ ; \ \ \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{b}{\pi\sin\theta(\gamma/E)}[b^2+(\gamma/E)]^{3/2} \ . \tag{20}$$

$$\text{Define } \cos\delta = \frac{b}{\sqrt{b^2+(\gamma/E)}} \ , \ \text{so } \sin\delta = \frac{(\gamma/E)^{1/2}}{\sqrt{b^2+(\gamma/E)}} \ , \ \text{and } \frac{d\sigma}{d\Omega} = \frac{\gamma/E}{\pi\sin\theta}\frac{\cos\delta}{\sin^4\delta} \ .$$

Now $\theta = \pi(1-\cos\delta) = 2\pi\sin^2(\delta/2)$ so $\sin^2(\delta/2) = \theta/(2\pi)$, $\cos^2(\delta/2) = 1-\theta/(2\pi)$, and putting the pieces together, we obtain the expression (17), whose behavior is sketched in Fig. 1.

This differential cross section goes to ∞ as $\theta \to 0$ and to $\gamma/(E\pi^2)$ as $\theta \to \pi$. The total cross section is infinite since the integral $\sigma_T = 2\pi\int_0^\pi \sin\theta d\theta(d\sigma/d\Omega)$ diverges at $\theta = 0$. For an attractive potential,

$$\theta(b) = \pi - 2b\int_0^{u_0}\frac{du}{\sqrt{1+(|\gamma|/E)u^2-b^2u^2}} = \pi\left(1-\frac{b}{\sqrt{b^2-(|\gamma|/E)}}\right) \ . \tag{21}$$

For $b \to \infty$, $\theta(b) \to 0$, while for $b \to \sqrt{|\gamma|/E}$, $\theta(b) \to \infty$. This means the particle executes an infinite spiral around the center of force. For all $b < \sqrt{|\gamma|/E}$, the particle is captured.

(4) (2 points) [FW 1.17, first part only]: A uniform beam of particles with energy E is scattered by an attractive central potential

$$V(r) = \begin{cases} 0 & r > a \\ -V_0 & r < a \end{cases} \tag{22}$$

Show that the orbit of a particle is identical with that of a light ray refracted by a sphere of radius a and index of refraction $n = [(E+V_0)/E]^{1/2}$.

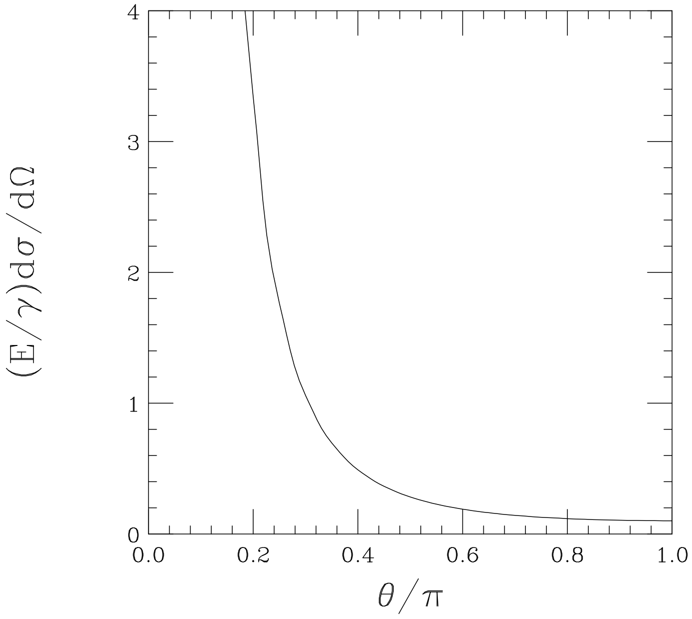

Figure 1: Scaled differential cross section for a repulsive $1/r^2$ potential.

Answer: For $b > a$, there is no scattering: $\theta(b) = 0$. For $b < a$ we have

$$\theta(b) = -\pi + 2b\int_0^{1/a} \frac{du}{\sqrt{1-u^2b^2}} + 2b\int_{1/a}^{u_0} \frac{du}{\sqrt{1+(V_0/E)-b^2u^2}} \ , \tag{23}$$

where $u_0 \equiv [1+(V_0/E)]^{1/2}/b$ and we have taken the particle incident from the left. The first integral is

$$2b\int_0^{1/a} \frac{du}{\sqrt{1-u^2b^2}} = 2\sin^{-1}(b/a) \ , \tag{24}$$

while the second is

$$\int_{1/a}^{u_0} \frac{du}{\sqrt{1+(V_0/E)-b^2u^2}} = \pi - 2\sin^{-1}\left[\frac{b}{na}\right] \ , \quad n \equiv \left(1+\frac{V_0}{E}\right)^{1/2} \ . \tag{25}$$

Thus, for $b < a$,

$$\theta(b) = 2\sin^{-1}\frac{b}{a} - 2\sin^{-1}\frac{b}{na} \ . \tag{26}$$

Compare this with scattering on a refractive sphere using Snell's Law. For $b < a$ a light ray hits the sphere making an angle $\theta_1 = \sin^{-1}(b/a)$ with the radius vector. It then travels inside the sphere in a straight line making an angle θ_2 with the radius vector. By Snell's Law, $n\sin\theta_2 = \sin\theta_1$, so $\theta_2 = \sin^{-1}(b/na)$ and the ray has been deflected by an angle $\phi = \theta_1 - \theta_2$. As it exits the sphere, the symmetry of the geometry implies that it is deflected by an equal angle ϕ, so the total deflection is $\theta = 2\phi = 2(\theta_1 - \theta_2)$ which is just Eq. (26).