

ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

PROBLEM SET #2 AND ANSWERS - DUE TUESDAY, OCTOBER 21

This problem set will be worth 10 points. Notation for problems: G = Goldstein *et al.*; PWJ: Porter W. Johnson's draft; FW = Fetter and Walecka. First number is chapter; second is problem number.

(1) (3 points). We showed in class that the motion of a body with mass m in a central potential $V(r)$ can be described by the angle

$$\theta(r) = \int^r \frac{\ell/r^2}{\sqrt{2m[E - V(r)] - \ell^2/r^2}} dr, \quad (1)$$

where $\ell = mr^2\dot{\theta}$ is the (conserved) orbital angular momentum. Consider an attractive power-law potential $V(r) = \lambda r^\alpha$, where $\lambda < 0$ for $-2 < \alpha < 0$ and $\lambda > 0$ for $0 < \alpha < \infty$. For simplicity set $m = 1$. Then

$$\theta(r) = \int^r \frac{1}{r^2 \sqrt{2[E - \lambda r^\alpha]/\ell^2 - 1/r^2}} dr. \quad (2)$$

Show using the substitution $u = r^{-\alpha/\bar{\alpha}}$, where $\bar{\alpha}$ is to be suitably defined, that the corresponding equation in u can be written

$$\theta(u) = -\frac{\bar{\alpha}}{\alpha} \int^{u=r^{-\alpha/\bar{\alpha}}} \frac{1}{u^2 \sqrt{2[-\lambda + Eu^{\bar{\alpha}}]/\ell^2 - 1/u^2}} du, \quad (3)$$

which is a central force problem with potential $-Eu^{\bar{\alpha}}$ and energy $-\lambda$. Find the relation between α and $\bar{\alpha}$ such that this relation holds. Show that $-2 < \alpha < 0$ implies $\infty > \bar{\alpha} > 0$ and that $\infty > \alpha > 0$ implies $-2 < \bar{\alpha} < 0$. Show in particular that the Kepler problem ($\alpha = -1$) is equivalent to the harmonic oscillator ($\bar{\alpha} = 2$) and vice versa.

Note: These relations were known to Newton for small oscillations about circular orbits. He originally solved the Kepler problem by first mapping it into the harmonic oscillator in two dimensions.

Answer: With $u^{\bar{\alpha}} = r^{-\alpha}$, we have $\bar{\alpha}u^{\bar{\alpha}-1}du = -\alpha r^{-(\alpha+1)}d\alpha$. We will choose $\bar{\alpha}$ so that the integral with respect to u looks like an integral for a different potential, with E and λ trading places so that the new potential is $-Eu^{\bar{\alpha}}$ and the new energy is $-\lambda$. We get

$$\theta(r) = -\frac{\bar{\alpha}}{\alpha} \int^{u(r)} \frac{du}{u^2 \sqrt{2(Eu^{-(2\bar{\alpha}/\alpha+2)} - \lambda u^{-(2\bar{\alpha}+2+\bar{\alpha})})/\ell^2 - 1/u^2}}. \quad (4)$$

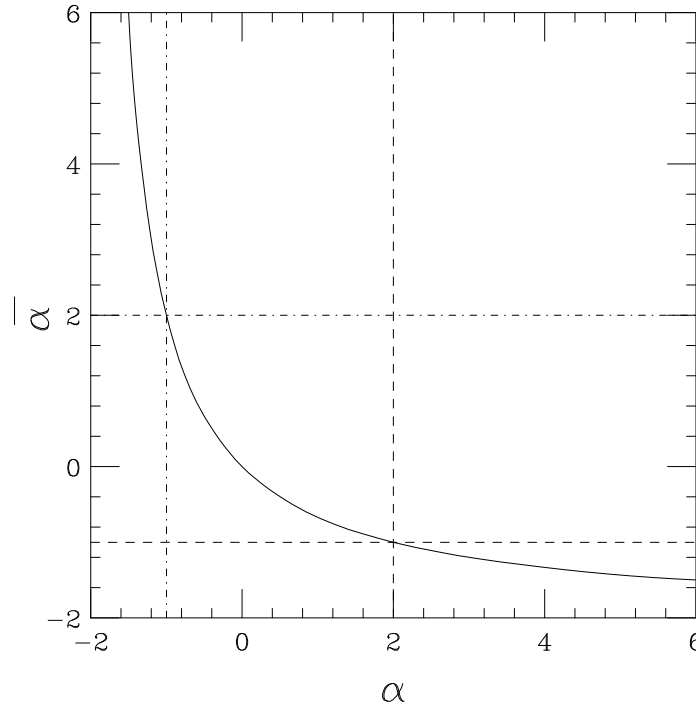
We want to make the λ term into a constant, so we choose

$$\frac{2\bar{\alpha}}{\alpha} + 2 + \bar{\alpha} = 0, \quad (2 + \bar{\alpha})(2 + \alpha) = 4. \quad (5)$$

Then the power of u multiplying E is $-(2\bar{\alpha}/\alpha + 2) = \bar{\alpha}$, and

$$\theta(r) = -\frac{\bar{\alpha}}{\alpha} \int^{u(r)} \frac{du}{u^2 \sqrt{2(Eu^{\bar{\alpha}} - \lambda)/\ell^2 - 1/u^2}}, \quad (6)$$

which indeed corresponds to a new potential $\bar{V}(u) = -Eu^{\bar{\alpha}}$, a new energy $\bar{E} = -\lambda$, and a new angle $\bar{\theta} = -\alpha\theta/\bar{\alpha}$. For some applications and details see A. K. Grant and J. L. Rosner, *Am. J. Phys.* **62**(4), 310 (1994). The relation between α and $\bar{\alpha}$ is shown in the figure. Note the symmetry between α and $\bar{\alpha}$. The dashed lines correspond to $\alpha = 2$, $\bar{\alpha} = -1$, while the dotdashed lines correspond to $\alpha = -1$, $\bar{\alpha} = 2$.



(2) (2 points) [FW 1.12]: The orbit of the planet Mercury has an eccentricity 0.206 and a period 0.241 year; moreover, the perihelion advances slowly at the rate of 43 seconds of arc per century. One possible explanation of this effect is that the potential energy around the Sun has the form $V = -(mMG/r)(1 + \alpha GM/rc^2)$, where α is a dimensionless constant and $MG/c^2 \equiv r_0 \approx 1.475$ km characterizes the Sun's gravitational field. Demonstrate that the resulting orbit indeed represents a precessing ellipse. Find the magnitude and sign of α needed to fit the observed data. (Related problems: G 3.21, 3.22.)

Answer: The expression for the angle $\theta(r)$ is

$$\theta(r) = \frac{1}{\sqrt{2m}} \int^r \frac{\ell dr/r^2}{\sqrt{E - \ell^2/(2mr^2) - V(r)}}. \quad (7)$$

Adding the term $\Delta V = -\alpha r_0 m M G / r^2$ to the lowest-order potential $V_0 = -(m M G) / r$ is equivalent to replacing

$$\frac{\ell^2}{2m} \rightarrow \frac{\ell'^2}{2m} = \frac{\ell^2}{2m} - \alpha r_0 m M G \quad \text{or} \quad \ell' = \ell - \frac{\alpha r_0 m^2 M G}{\ell} \quad (8)$$

in the denominator of (7), whereas ℓ in the numerator is not changed. If we also had ℓ' in the numerator, we would have a closed elliptical orbit. Hence the precession angle per orbit is

$$\Delta\theta = 2\pi \left[\frac{\ell}{\ell'} - 1 \right] = \frac{2\pi\alpha r_0 m^2 M G}{\ell^2} . \quad (9)$$

We need to express $\ell^2 / (m^2 M G)$ as a length, given the parameters of Mercury's orbit. This is just the parameter p of the ellipse $(p/r) = 1 + e \cos \theta$:

$$p = \frac{\ell^2}{m^2 M G} = a(1 - e^2) , \quad (10)$$

where by Kepler's Third Law $a/a_\oplus = (0.241 \text{ yr}/1 \text{ yr})^{2/3} = 0.387$, so $p = 0.387 \text{ AU}(1 - 0.206^2) = 0.371 \text{ AU} = 5.55 \times 10^7 \text{ km}$. Here we have used $1 \text{ AU} = a_\oplus = 1.496 \times 10^8 \text{ km}$. Then

$$\Delta\theta = 2\pi\alpha \frac{1.475 \text{ km}}{5.55 \times 10^7 \text{ km}} = (1.67 \times 10^{-7})\alpha . \quad (11)$$

Now 43 seconds of arc per century is 0.0119° per century $= 2.085 \times 10^{-6} / \text{yr} = 5.024 \times 10^{-7}$ in a "Mercury year" (0.241 yr). So apparently $\alpha = 3$. (See Weinberg, *Gravitation and Cosmology*, pp. 194-201.)

(3) (4 points) [G 2.6]: Find the Euler-Lagrange equation describing the brachistochrone curve for a particle moving *inside* a spherical Earth of uniform mass density. Obtain a first integral for this differential equation by analogy to the Jacobi integral h [Goldstein's notation; PWJ calls it J]. With the help of this integral, show that the desired curve is a hypocycloid (the curve described by a point on a circle rolling on the inside of a larger circle). (Hint: take the larger circle to have the radius R of the Earth.) Obtain an expression for the time of travel along the brachistochrone between two points on Earth's surface. How long would it take to go from New York to Los Angeles (assumed to be 4800 km apart on the surface) along a brachistochrone tunnel (assuming no friction) and how far below the surface would the deepest point of the tunnel be? [Note: PWJ gives many hints in his Problem 6.3 but I believe he has the sign of $V(r)$ wrong.]

Answer: Let r be the distance from the center of the Earth, and \vec{v} be the local velocity of the mass m . The kinetic energy is $T = \frac{1}{2} m v^2$, while the potential energy (taken to be zero at $r = R$, where $R \simeq 6400 \text{ km}$ is the radius of the Earth), is

$$V = \frac{1}{2} m g R \left(\frac{r^2}{R^2} - 1 \right) . \quad (12)$$

This may be found by calculating the gravitational attraction at a radius r as due entirely to the mass *within* a sphere of radius r , and assuming all the mass is concentrated at the origin (a result known to Newton). Taking the mass m to have zero initial speed $v = 0$ at the surface of the Earth, one has

$$T + V = 0 \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}mgR\left(\frac{r^2}{R^2} - 1\right) = 0 \Rightarrow v = \sqrt{gR\left(1 - \frac{r^2}{R^2}\right)}. \quad (13)$$

We wish to minimize

$$t = \int dt = \int \frac{ds}{ds/dt} = \int \frac{dr\sqrt{1 + r^2\theta'^2}}{\sqrt{gR(1 - r^2/R^2)}} \quad (14)$$

since $ds^2 = dr^2 + r^2d\theta^2 = dr\sqrt{1 + r^2\theta'^2}$ with $\theta' \equiv d\theta/dr$. The variational equation in $\theta(r)$ is then

$$\frac{d}{d\theta'} \frac{df}{d\theta'} - \frac{df}{d\theta} = 0, \quad f \equiv \left[\frac{1 + r^2\theta'^2}{gR(1 - r^2/R^2)} \right]^{1/2}. \quad (15)$$

Now f is independent of θ , so $df/d\theta' = \text{const.}$ Thus

$$\frac{df}{d\theta'} = \frac{2r^2\theta'}{(1 + r^2\theta'^2)^{1/2} \left(gr \left[1 - \frac{r^2}{R^2} \right] \right)^{1/2}} = \text{const.} \Rightarrow \frac{r^2\theta'}{(1 + r^2\theta'^2)^{1/2} v} = \text{const.} \quad (16)$$

The condition $v = 0$ at $t = 0$ then requires $\theta' = 0$ at $t = 0$. We now have to show that this is a hypocycloid. This equation is equivalent to

$$\frac{r^2 d\theta/dr}{(ds/dr)(1 - r^2/R^2)^{1/2}} = \alpha \text{ (const.)} \Rightarrow \frac{d\theta}{ds} = \frac{\alpha}{r^2} \left(1 - \frac{r^2}{R^2} \right)^{1/2}. \quad (17)$$

Let $x = r \sin \theta$, $y = r \cos \theta$ (so that $r = R$, $\theta = 0$ corresponds to the North Pole $x = 0$, $y = R$). Let a small circle of radius a roll on a large circle of radius R (the radius of the Earth). Let the point of contact make an angle ϕ with the vertical (y) axis. A point on the small circle starts out at $x = 0$, $y = R$ and rotates counterclockwise with respect to the point of contact by an angle $\psi = (R/a)\phi$, so it makes an angle of $\psi - \phi = \phi[(R/a) - 1]$ with respect to the vertical.

The point of contact and a point on the small circle coincide at $\phi = 0$ and again at $\phi_0 = 2\pi a/R$. (You may take the case $a/R = 1/4$, which implies $\phi_0 = \pi/2$, and convince yourself that this is reasonable with a sketch.) So, to describe a hypocycloid from $(r = R, \theta = 0)$ to $(r = R, \theta = \theta_0)$ we choose $a = (\theta_0/2\pi)R$. One then calculates that the point on the small circle has the coordinates

$$x = (R - a) \sin \phi - a \sin \left[\phi \left(\frac{R}{a} - 1 \right) \right], \quad (18)$$

$$y = (R - a) \cos \phi + a \cos \left[\phi \left(\frac{R}{a} - 1 \right) \right] \quad (19)$$

and hence, after a short calculation, that

$$ds = (dx^2 + dy^2)^{1/2} = 2(R - a) \sin \frac{\phi R}{2a} d\phi . \quad (20)$$

Moreover, $\tan \theta = x/y$, so (always considering points on the hypocycloid) we may write

$$d(\tan \theta) = \sec^2 \theta d\theta = \frac{ydx - xdy}{y^2} = \frac{ydx - xdy}{r^2 \cos^2 \theta} , \quad \text{or} \quad (21)$$

$$d\theta = \frac{ydx - xdy}{r^2} = \frac{2(R - a)(R - 2a) \sin^2(\phi R/2a)}{r^2} d\phi . \quad (22)$$

Thus

$$\frac{d\theta}{ds} = \frac{R - 2a}{r^2} \sin \frac{\phi R}{2a} . \quad (23)$$

Also $r^2 = x^2 + y^2 = R^2 - 4a(R - a) \sin^2(\phi R/2a)$ or

$$\sin \frac{\phi R}{2a} = \left[\frac{R^2 - r^2}{4a(R - a)} \right]^{1/2} . \quad (24)$$

Thus a hypocycloid indeed satisfies the condition

$$\frac{d\theta}{ds} = \frac{\text{const.}}{r^2} \left[1 - \frac{r^2}{R^2} \right]^{1/2} . \quad (25)$$

For the path from New York to Los Angeles, a distance of 4800 km, since $R = 6378$ km, $\theta_0 = 0.753$, and

$$\frac{a}{R} = \frac{\theta_0}{2\pi} = 0.120 \Rightarrow a = 764 \text{ km} . \quad (26)$$

The maximum depth in the Earth is $2a \simeq 1528$ km. The time of travel is

$$t = \int \frac{ds}{v} = \int 2(R - a) \sin \frac{\phi R}{2a} d\phi \left[gr \left(1 - \frac{r^2}{R^2} \right) \right]^{-1/2} . \quad (27)$$

Substituting Eq. (24), we find that the integral is just proportional to $d\phi$:

$$t = \phi \sqrt{\frac{R(R - a)}{ga}} = \theta_0 \sqrt{\frac{R}{g}} \sqrt{\frac{2\pi}{\theta_0} - 1} , \quad (28)$$

where we have used $R/a = 2\pi/\theta_0$. For $\theta = \pi$ (travel through the Earth's center to the other side) we get the familiar result $t = \pi\sqrt{R/g} \simeq 42$ min. Here we have used $R = 6378$ km and $g = 9.8$ m/s². For $\theta_0 \simeq 3/4$ radian,

$$t = (42 \text{ min}) \left(\frac{\theta_0}{\pi} \right) \sqrt{\frac{2\pi}{\theta_0} - 1} \simeq 27 \text{ min} . \quad (29)$$

(4) (1 point) [G 3.11]: Two particles move about each other in circular orbits under the influence of gravitational forces, with a period τ . Their motion is suddenly stopped

at a given instant of time, and they are then released and allowed to fall into one another. Prove that they collide after a time $\tau/4\sqrt{2}$.

Answer: Start out with $\dot{r} = 0$ at $r = R$, the initial relative separation. The total energy is then just the potential energy $V = -Gm_1m_2/R$. We have to integrate the energy equation

$$T + V = \frac{1}{2}\mu\dot{r}^2 - \frac{Gm_1m_2}{r} = \frac{Gm_1m_2}{R}, \quad (30)$$

where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass. Defining $M = m_1 + m_2$, we have to solve

$$\dot{r}^2 = 2GM \left(\frac{1}{r} - \frac{1}{R} \right) \quad (31)$$

which can be done by elementary integration. Taking the correct sign for the square root since r is decreasing when t is increasing, we find

$$dt = \frac{-dr}{\sqrt{2GM \left(\frac{1}{r} - \frac{1}{R} \right)}} = \frac{-\sqrt{Rr}dr}{\sqrt{2GM(R-r)}}. \quad (32)$$

The integral must be taken from $r = R$ to $r = 0$, or, defining $x = 1 - (r/R)$, we find

$$t = \sqrt{\frac{R^3}{2GM}} \int_0^1 dx (1-x)^{1/2} x^{-1/2} = \sqrt{\frac{R^3}{2GM}} \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)}. \quad (33)$$

Evaluating the Γ functions by noting that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$, while $\Gamma(2) = 1! = 1$, we find

$$t = \frac{\pi}{2} \sqrt{\frac{R^3}{2GM}} = \frac{\tau}{4\sqrt{2}}. \quad (34)$$