ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

PROBLEMS DUE TUESDAY, OCTOBER 14 - ANSWERS

This problem set will be worth 12 points and contains an additional *optional* problem (#5). Subsequent problem sets will be worth 10 points. Notation for problems: G = Goldstein *et al.*; PWJ: Porter W. Johnson's draft; FW = Fetter and Walecka. First number is chapter; second is problem number.

(1) [G, 1.12]: The *escape velocity* of a particle on Earth is the minimum velocity required at Earth's surface in order that the particle can escape from Earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy, show that the escape velocity for Earth, ignoring the presence of the Moon, is 11.2 km/s (1 point).

Answer: The force of gravity at the Earth's surface ($r = r_\oplus$) on an object of mass m, $F = -mg$, where $g = 9.8$ m/s^2, for a gravitational potential $V(r) = -\alpha/r$, is equal to $-\partial V/\partial r = -\alpha/r_\oplus^2$, so that $\alpha = gmr_\oplus^2$ and hence the rocket needs a kinetic energy of $(1/2)mv_{\rm E}^2 = -V(r_\oplus) = gmr_\oplus$ to escape, where the escape velocity $v_{\rm E}$ is

$$v_{\rm E} = \sqrt{2gr_\oplus} = 11.2 \text{ km/s} \ . \tag{1}$$

We have used the Earth's radius $r_\oplus = 6378$ km [See the Particle Data Group's compilation of physical constants, e.g., J. Phys. G **33**, 1 (2006)].

(2) [G, 1.13]: Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg \ , \tag{2}$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300 (2 points).

Answer: Let $v = v_R^\oplus$ be the velocity of the rocket in the Earth's reference frame and $p = mv$ be its momentum. Let $v_{\rm gas}^\oplus$ be the velocity of the gas escaping from the rocket in the Earth's reference frame. Then

$$\frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = -v_{\rm gas}^\oplus\frac{dm}{dt} - mg \ . \tag{3}$$

Bearing in mind that the velocity of the gas in the *rocket's* rest frame is $v' = v + v^{\oplus}_{\text{gas}}$, we obtain Eq. (2). Noting that $m = m_0 - \alpha t$, where m_0 is the initial mass and $\alpha = m_0/(60\text{s})$, we find

$$(m_0 - \alpha t)\frac{dv}{dt} = -\alpha v' - (m_0 - \alpha t)g \ , \quad \text{or} \tag{4}$$

$$\frac{dv}{dt} = -\frac{\alpha v'}{m_0 - \alpha t} - g \ . \tag{5}$$

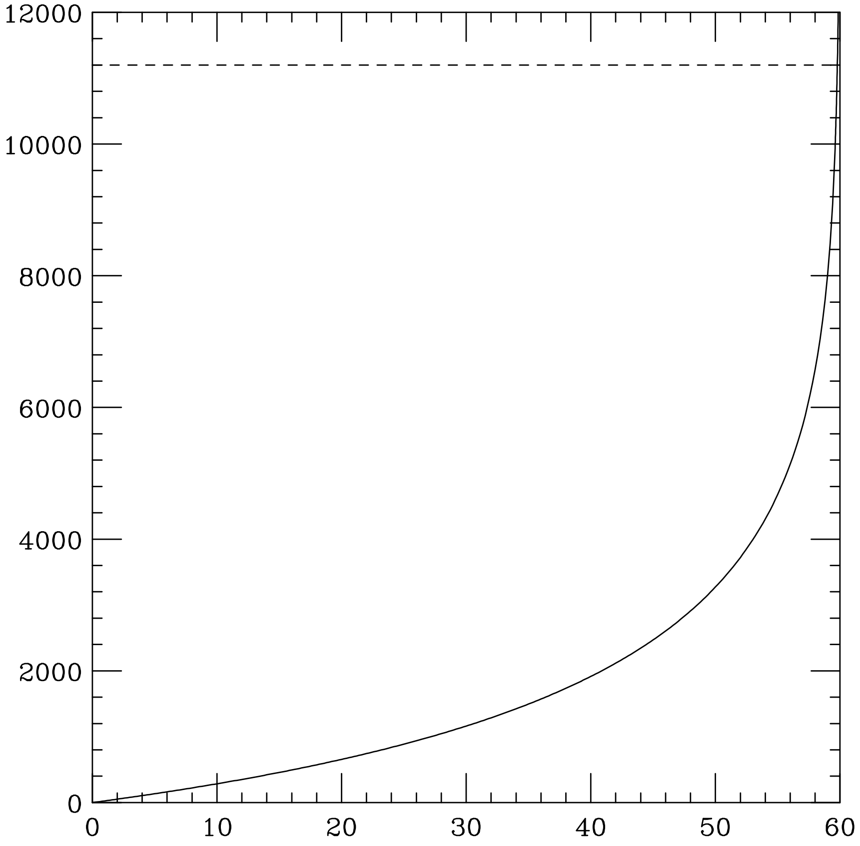
Integrating this once with respect to t, we find

$$v(t) = v(0) - gt + v' \ln\left(\frac{m_0 - \alpha t}{m_0}\right) \ . \tag{6}$$

For a rocket initially at rest with $v(0) = 0$ and with the exhaust gas having a velocity $v' = -2.1$ km/s in the rocket's rest frame, we find that the escape velocity $v_{\text{E}} = 11.2$ km/s is reached when

$$v_{\text{E}} = 11.2 \text{ km/s} = -gt - 2.1 \text{ (km/s)} \ln\left(1 - \frac{t}{60 \text{ s}}\right) \ . \tag{7}$$

The function on the right-hand side (in m/s) is plotted as a function of time (in seconds) in the figure. The escape velocity (dashed line) is reached when $t = 59.78$ seconds, at which time the empty rocket weighs only 1/274 of its initial weight.



(3) [PWJ, 1.2]: *Fermat's Principle* is the statement that a ray of light in a medium with a variable index of refraction will follow the path that requires the shortest travel time. For two-dimensional travel in the x–y plane, show that such a path is obtained by minimizing the quantity

$$cT = \int_a^b n(x,y)\sqrt{1+y'^2}dx \ . \tag{8}$$

Note: In a medium with index of refraction n, the speed of light is c/n.

For the particular case $n = \kappa/y$ for $y > 0$, show that the rays of light travel along paths of the form

$$(x - x_0)^2 + y^2 = a^2 \ , \tag{9}$$

which are semi-circles centered upon the x-axis (2 points).

Answer: The instantenous unit of arc length is $ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+y'^2}$, and the time it takes for a light ray to traverse this unit of arc length is $dt = (n/c)dx$. Integrating, we obtain Eq. (8).

When $n = \kappa/y$, the time for the light ray to travel between two points is

$$T = \int_a^b dx \frac{\kappa}{cy}\sqrt{1+y'^2} \ , \tag{10}$$

so that the first integral of the corresponding Euler-Lagrange equation is $1/[y(1+y'^2)] = \text{const}$. Equivalently, $1 + y'^2 = a^2/y^2$, where a is some constant, and this may be written as

$$\frac{dy}{\sqrt{(a/y)^2-1}} = \frac{ydy}{\sqrt{a^2-y^2}} = dx \ . \tag{11}$$

With the change of variables $v = a^2 - y^2$ this may be written as $-dv/\sqrt{v} = 2dx$ so that $\sqrt{v} = -(x - x_0)$ and hence $(x - x_0)^2 + y^2 = a^2$.

(4) [PWJ, 1.5]: *String Hanging over Table*: A uniform string of length ℓ is draped over the edge of a horizontal table, initially at rest with a piece x_0 extending over the edge. Show that the amount of string hanging over the edge at time t, $x(t)$, satisfies the equation $\ell\ddot{x} = gx$, so that (1 point)

$$x(t) = x_0 \cosh\sqrt{\frac{g}{\ell}}t \ . \tag{12}$$

Answer: Let μ be the mass per unit length. The kinetic energy is $T = (1/2)\mu\ell\dot{x}^2$, while the potential energy, taking account of the fact that the piece of string hanging over the edge of the table has length μx and center of mass $x/2$, is $V = -\mu g x^2/2$. Thus the Lagrangian is

$$L = \mu\ell\frac{\dot{x}^2}{2} + \frac{\mu x^2}{2}g \ . \tag{13}$$

Lagrange's equation thus implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \mu \ell \ddot{x} = \frac{\partial L}{\partial x} = \mu g x \ . \tag{14}$$

The solution to this equation which satisfies $\dot{x}(0) = 0$ is Eq. (12).

(5) [PWJ, 1.6]: *Falling Chain*: A chain of length ℓ lies pushed together at the edge of a table, except for a piece of length x_0 which hangs over it. Show that the length x over the edge at time t satisfies the equation $d(x\dot{x})/dt = gx$. Therefore, show that the corresponding speed of the falling chain is

$$v^2 = \frac{2g}{3}\left(x - x_0^3/x^2\right) \ . \tag{15}$$

Show that, just as the chain falls off the table,

$$v^2 = \frac{2g\ell}{3}\left(1 - x_0^3/\ell^3\right) \ . \tag{16}$$

This problem originally comes from Arnold Sommerfeld's *Lectures on Theoretical Physics, Volume I — Mechanics*, Academic Press, p. 241. The claim is that energy is not conserved because it is dissipated as each link successively falls off the table. The problem is *optional.* I would welcome a simple motivation of the equation of motion. If one ignores this dissipation, one gets a different result.

Answer: Let μ be the mass per unit length, as in the previous problem. The potential energy is the same as in the previous problem, $V = -\mu g x^2/2$. The kinetic energy is now $T = \mu x \dot{x}^2/2$, as there is a mass μx moving with velocity $\dot{x}$. If one were to assume energy conservation, the Lagrangian would be

$$L = \mu x \frac{\dot{x}^2}{2} + \frac{\mu x^2}{2} g \ , \tag{17}$$

and the constancy of $\dot{x}(\partial L/\partial \dot{x}) - L$ would imply that $\dot{x}^2 - xg = \text{const}/x$, or, applying the condition that $\dot{x} = 0$ at $x = x_0$, one would have

$$\dot{x}^2 = v^2 = g\left(x - \frac{x_0^2}{x}\right) \ . \tag{18}$$

However, as each link is dragged off the table and set in motion, Porter Johnson argues that energy is not conserved. This appears reasonable since an inelastic collision between a moving link and a stationary one is required. Momentum is still conserved. The momentum of the moving part of the chain is $p = \mu x \dot{x}$, while the force is $F = -(\partial V/\partial x) = \mu g x$. Equating $dp/dt = F$, one gets the equation

$$\frac{d(x\dot{x})}{dt} = \dot{x}^2 + x\ddot{x} = gx \ . \tag{19}$$

To convert this to an equation for $v^2 \equiv \dot{x}^2$, one notes that $\ddot{x} = dv/dt = (dv/dx)(dx/dt) = v(dv/dx)$ and, furthermore,

$$xv^2 + x^2 v\frac{dv}{dx} = \frac{1}{2}\frac{d}{dx}(v^2x^2) = gx^2 \ . \tag{20}$$

Integrating this equation with respect to x and applying the condition $v(x_0) = 0$, we obtain Eq. (15).

(6) [PWJ, 1.8]: *Minimum Area under Curve*: Show that for a planar curve $y(x)$ containing two points, $(a, y(a))$ and $(b, y(b))$, with a fixed length

$$\ell = \int_a^b \sqrt{1 + y'^2(x)}dx \ , \tag{21}$$

the minimum area under the curve,

$$A = \int_a^b y(x)dx \tag{22}$$

occurs for a circular arc passing through these points. Show, furthermore, that the ratio of the Euclidean distance between points d and the length ℓ of the curve are related to the parameter ρ through the transcendental equation

$$\frac{d}{\ell} = \frac{\sin\rho}{\rho} \tag{23}$$

where the radius R of the arc is $r = \ell/(2\rho)$. There is one solution ρ of Eq. (23) on the interval $0 \leq \rho \leq \pi$, since $d < \ell$ and the function on the right of (23) is monotonically decreasing in ρ. The radius R may then be determined. Show that the area under the curve is (2 points)

$$A = \frac{y(b) + y(a)}{2}(b - a) - R^2(\rho - \sin\rho\cos\rho) \ . \tag{24}$$

Answer: Using a Lagrange multiplier λ, we minimize the combination

$$A - \lambda\ell = \int_a^b dxF(y, y') \ , \quad F(f, y') \equiv y - \lambda\sqrt{1 + y'^2} \ . \tag{25}$$

As F is independent of x, the combination $y'(\partial F/\partial y') - F$ must be a constant, leading to the result

$$\frac{\lambda}{\sqrt{1 + y'^2}} - y = \beta \ . \tag{26}$$

The integration proceeds here in a manner similar to that in Problem [PWJ, 1.2]. After some algebra, Eq. (26) may be written

$$\frac{(y + \beta)dy}{\sqrt{\lambda^2 - (y + \beta)^2}} = dx = -\frac{dv}{2\sqrt{v}} \ , \tag{27}$$

where we have made the substitution $v = \lambda^2 - (\beta + y)^2$. Integrating this equation we find $-\sqrt{v} = x + \alpha$, or, squaring,

$$\lambda^2 - (y + \beta)^2 = (x + \alpha)^2 \ , \tag{28}$$

which is a circular arc of radius $R = \lambda$, centered at $x = -\alpha$ and $y = -\beta$. Its length is

$$\ell = \int_a^b dx\sqrt{1 + y'^2} = \int_a^b dx \frac{R}{y + \beta} = \int_a^b dx \frac{R}{\sqrt{R^2 - (x + \alpha)^2}} \tag{29}$$

This integral may be performed by substituting $x + \alpha = R\sin\phi$ (so that $y + \beta = R\cos\phi$). Then $dx = R\cos\phi\ d\phi$, and $\ell = R[\phi(b) - \phi(a)]$, where $\phi(a)$ and $\phi(b)$ are determined by the requirement that the circle pass through the points $(a, y(a))$ and $(b, y(b))$.

Define $\rho = [\phi(b) - \phi(a)]/2$. The Euclidean distance d between the points $(a, y(a))$ and $(b, y(b))$ is $d = 2R\sin\rho$, while the length of the arc is $\ell = 2R\rho$. Taking the quotient, one has $\sin\rho/\rho = d/\ell$, which always has a (non-zero) solution as long as $d < \ell$.

The area under the curve is the difference between the area A_1 under the straight line joining $(a, y(a))$ and $(b, y(b))$ and the area A_2 between this straight line and the curve. We have $A_1 = (b - a)[y(a) + y(b)]/2$, while $A_2 = \rho R^2 - (R\cos\rho)(R\sin\rho)$. Then $A = A_1 - A_2$ gives the result (24).

(7) [FW, 1.2] A uniform spool of mass M and diameter d rests on a frictionless table. A massless string wrapped around the spool is attached to a weight m which hangs over the edge of the table. If the spool is released from rest when its center of mass is a distance ℓ from the edge of the table, what is the velocity of the weight m when the center of mass of the spool reaches the edge of the table? (2 points)

Answer: Let the density of the spool be ρ and its height h. Its mass is $M = \pi(d/2)^2 h\rho$ while its moment of inertia is

$$I = 2\pi h\rho \int_0^{d/2} r^2 \cdot r\ dr = Md^2/8\ . \tag{30}$$

Let x be the horizontal displacement of the spool on the table, and y be the distance the weight m hangs down over the table. Then the kinetic energy of the system is

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2\ , \tag{31}$$

while the potential energy is $V = -mgy$. The variables are connected by the constraint $y - x = d\theta/2$. This may be imposed via a Lagrange multiplier, but it is just as simple to substitute for θ, leading to the Lagrangian

$$L = \frac{1}{2}\left(m + \frac{1}{2}M\right)\dot{y}^2 - \frac{1}{2}M\dot{x}\dot{y} + \frac{3}{4}M\dot{x}^2 + mgy\ . \tag{32}$$

As L is independent of x, we have $\partial L/\partial\dot{x} = 0$, and imposing the condition that $\dot{x}(0) = \dot{y}(0) = 0$, we can integrate the resulting equation to find $\dot{x} = \dot{y}/3$. This may be substituted back into the Lagrangian to give

$$L = \frac{1}{2}\left(m + \frac{1}{3}M\right)\dot{y}^2 + mgy\ . \tag{33}$$

The total energy

$$E = \frac{1}{2}\left(m + \frac{1}{3}M\right)\dot{y}^2 - mgy \tag{34}$$

is conserved, and since $y(0) = \dot{y}(0) = 0$, $E = 0$. Because $\dot{y} = 3\dot{x}$ and both x and y are initially zero, the weight falls a distance 3ℓ when the spool travels a distance ℓ to the edge of the table. The velocity $\dot{y}$ of the weight thus satisfies

$$3mg\ell = \frac{1}{2}\left(m + \frac{1}{3}M\right)\dot{y}^2\ , \quad \dot{y} = \sqrt{\frac{6g\ell}{1 + \frac{M}{3m}}}\ . \tag{35}$$

(8) [FW, 3.5]: A massless inextensible string passes over a pulley which is a fixed distance above the floor. A bunch of bananas of mass m is attached to one end A of the string. A monkey of mass M is initially at the other end B. The monkey climbs the string, and its displacement $d(t)$ with respect to the end B is a *given* function of time. The system is initially at rest, so that the initial conditions are $d(0) = \dot{d}(0) = 0$. Introduce suitable generalized coordinates and calculate the lagrangian of the system in terms of these coordinates. Show that the equation of motion governing the height Z of the monkey above the floor is

$$(m + M)\ddot{Z} - m\ddot{d} = (m - M)g\ . \tag{36}$$

Integrate the equation to find the subsequent motion. In the special case that $m = M$, show that the bananas and the monkey rise through equal distances so that the vertical separation between them is constant (2 points). What is the condition for the monkey to be able to reach the bananas?

Answer: Let the pulley be a height h above the floor, the length of the string be ℓ, and the height of the bananas above the floor be $y(t)$. These quantities are related, together with the height $Z(t)$ of the monkey above the floor, by the constraint

$$\ell - d(t) + Z(t) + y(t) = 2h\ , \tag{37}$$

which will be imposed using a Lagrange multiplier. The kinetic and potential energies are

$$T = \frac{1}{2}M\dot{Z}^2 + \frac{1}{3}m\dot{y}^2\ , \quad V = MgZ + mgy\ , \tag{38}$$

so one applies Lagrange's equations to the quantity

$$L - \lambda(\text{constr.}) = \frac{1}{2}M\dot{Z}^2 + \frac{1}{3}m\dot{y}^2 - MgZ - mgy - \lambda(\ell - d + Z + y - 2h)\ . \tag{39}$$

The resulting equations of motion are

$$M\ddot{Z} = -Mg - \lambda\ , \quad m\ddot{y} = -mg - \lambda\ , \tag{40}$$

so that $M(\ddot{Z} - \ddot{y}) = -(M - m)g$. Furthermore (from the constraint) $\ddot{y} = \ddot{d} - \ddot{Z}$, so, substituting for $\ddot{y}$, one finds Eq. (36).

For the case $m = M$, one has $\ddot{Z} = \ddot{y} = \ddot{d}/2$. Since the system is assumed to be initially at rest with $d(0) = \dot{d}(0) = \dot{Z}(0) = \dot{y}(0) = 0$, integration shows that the monkey and the bananas then must rise through equal distances. More generally, integrating Eq. (36), one finds (with the above initial conditions)

$$(M + m)\dot{Z}(t) = (m - M)gt + m\dot{d}(t) \ . \tag{41}$$

If $\dot{d}(t)$ approaches a constant, the first term eventually dominates. For $m > M$, $\dot{Z}$ increases, while for $m < M$, $\dot{Z}$ decreases. Integrating once more, we find

$$(M+m)[Z(t)-Z(0)] = \frac{1}{2}(m-M)gt^2+md(t) = \frac{1}{2}(m-M)gt^2+m[Z(t)+y(t)-2h+\ell] \ . \tag{42}$$

If we define $Z(0) = 0$, then $y(0) = 2h - \ell$. Hence we have

$$MZ(t) = \frac{1}{2}(m - M)gt^2 + m[y(t) - 2h + \ell] \ . \tag{43}$$

Assume $y(0) > 0$ so that the bananas are initially above the monkey. If $m < M$, $Z(t) < y(t) - 2h + \ell$, and the monkey never gets the bananas.