ADVANCED CLASSICAL MECHANICS

Physics 316 - Fall Quarter, 2008 - University of Chicago

MIDQUARTER EXAMINATION - THURSDAY, OCTOBER 30 - ANSWERS

This examination contains three problems, each worth 10 points.

(1) (10 points) [FW 3.6]: An inextensible massless string of length l passes through a hole in a frictionless table. A point mass m at one end moves on the table and a point mass m hangs from the other end.

(a) Write the Lagrangian for this system.

Answer: Let the length of the string on the table be r, and the length below the table be z, with $l = r + z$. Let the angle the string on the table makes with some fixed direction be denoted by θ. The kinetic and potential energies are then

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \quad , \quad V = -mgz \; . \tag{1}$$

or, substituting $z = l - r$, the Lagrangian $L = T - V$ becomes

$$L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mg(l - r) \; . \tag{2}$$

(b) Under what condition will the hanging mass remain stationary?

Answer: The absence of the variable θ in L means that angular momentum is conserved: $mr^2\dot{\theta} = \ell =$ const. If $\dot{\theta} = \omega_0 =$ const., then r is also constant, $r = \sqrt{\ell/(m\omega_0)}$. Now, the r equation based on the Lagrangian is

$$2m\ddot{r} - mr\dot{\theta}^2 + mg = 0 \; , \tag{3}$$

so if $\ddot{r} = 0$, the equilibrium value r_0 must be $r_0 = g/\omega_0^2$.

(c) Starting from the situation in part (b), the hanging mass is pulled down slightly and released. State clearly what is conserved during this process.

Answer: When the weight is pulled down and released, one adds energy to the system, so energy is not conserved, but angular momentum is still conserved.

(d) Compute the subsequent motion of the hanging mass.

Answer: Eq. (3) implies

$$2\ddot{r} = r\dot{\theta}^2 - g = r\left(\frac{\ell}{mr^2}\right)^2 - g = \frac{\ell^2}{m^2r^3} - g \; , \tag{4}$$

Expand the right-hand about its equilibrium value of zero at $r = r_0$:

$$\frac{\partial}{\partial r}\left(\frac{\ell^2}{m^2r^3}\right)|_{r=r_0} = -\frac{3\ell^2}{m^2r_0^4} = -\frac{3g}{r_0} \;\Rightarrow\; \ddot{r} = -\frac{3g}{2r_0}\,(r - r_0) \; , \tag{5}$$

so the hanging mass executes simple harmonic motion with frequency $\Omega = \sqrt{3g/(2r_0)}$.

(2) (10 points): Two pendula each of length l and mass m are coupled by a spring whose force law is $F = -kx - a\dot{x}$. Discuss the normal modes for small oscillations and calculate the eigenfrequencies assuming a time dependence of the form $e^{-i\omega t}$. (Hint: discuss the undamped case first.)

Answer: In the absence of damping, leaving aside constants, the Lagrangian for small oscillations may be written

$$L = \frac{1}{2}ml^2(\dot{\theta_1}^2 + \dot{\theta_2}^2) - \frac{mgl}{2}(\theta_1^2 + \theta_2^2) - \frac{1}{2}kl^2(\theta_1 - \theta_2)^2 \ , \tag{6}$$

where θ_1 and θ_2 are the angles which the pendula make with the vertical direction. The corresponding equations of motion are

$$ml^2\ddot{\theta}_1 = -mgl\theta_1 - kl^2(\theta_1 - \theta_2) \ , \quad ml^2\ddot{\theta}_2 = -mgl\theta_2 - kl^2(\theta_2 - \theta_1) \tag{7}$$

for small oscillations. The normal modes are

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \quad \omega_2 = \frac{g}{l} \ , \tag{8}$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ , \quad \omega_2 = \frac{g}{l} + \frac{2k}{m} \ . \tag{9}$$

With the damping term, it is not straightforward to write a Lagrangian. However, the equations of motion themselves are, for small oscillations,

$$ml^2\ddot{\theta}_1 = -mgl\theta_1 - al(\dot{\theta}_1 - \dot{\theta}_2) - kl^2(\theta_1 - \theta_2) \ , \tag{10}$$

$$ml^2\ddot{\theta}_2 = -mgl\theta_2 - al(\dot{\theta}_2 - \dot{\theta}_1) - kl^2(\theta_2 - \theta_1) \ . \tag{11}$$

With $\theta_1 = c_1 e^{-i\omega t}$ and $\theta_2 = c_s e^{-i\omega t}$, we have

$$-m\omega^2 c_1 = -(mg/l)c_1 + i\omega a(c_1 - c_2) - k(c_1 - c_2) \ , \tag{12}$$

$$-m\omega^2 c_2 = -(mg/l)c_2 + i\omega a(c_2 - c_1) - k(c_2 - c_1) \ . \tag{13}$$

Taking the sum of these equations, we find an equation for $c_1 + c_2$ (i.e., a mode with $c_1 = c_2$) whose solution is unaffected by a or k and has $\omega^2 = g/l$. Taking the difference, we find an equation for $c_1 - c_2$ (i.e., a mode with $c_1 = -c_2$) whose solution requires ω to satisfy

$$\omega^2 = \frac{g}{l} + 2\left[\frac{k}{m} - \frac{i\omega a}{m}\right] \ ; \ \omega = -\frac{ia}{m} \pm \sqrt{\frac{g}{l} + 2\frac{k}{m} - \left(\frac{a}{m}\right)^2} \ . \tag{14}$$

The mode with $c_1 = c_2$ is undamped, because the two pendula swing in phase. The mode with $c_1 = -c_2$ is exponentially damped as a function of time because of the $-ia/m$ term. (Its decay is analogous to the decay of the short-lived neutral kaon into the $\pi\pi$ state.) The undamped frequency $\omega^2 = (g/l) + (2k/m)$ of this mode is also shifted by the damping term. Depending on whether $(a/m)^2$ is less than, equal to, or greater than $(g/l) + (2k/m)$, we will have an underdamped, critically damped, or overdamped solution for this mode.

(3) (10 points): Consider a nearly circular orbit in a potential $V(r) = \lambda r^\alpha$, with $\lambda\alpha > 0$. By examining the dependence of $u = 1/r$ on θ, show that

$$\frac{d^2u}{d\theta^2} = -u + \frac{m\lambda\alpha u^{-(\alpha+1)}}{\ell^2} \ , \tag{15}$$

where ℓ is the orbital angular momentum. Evaluate the equilibrium value $u = u_0$ and by expanding about this value, $u = u_0 + \xi$, show that

$$\frac{d^2\xi}{d\theta^2} = -(\alpha+2)\xi \ \text{ for small } \xi \ . \tag{16}$$

Interpret this result in terms of shapes of orbital patterns.

Answer: We consider a portion of the orbit for which $d\theta/dr > 0$. Since

$$\frac{d\theta}{dr} = \frac{1}{2m}\frac{\ell/r^2}{\sqrt{E - \ell^2/(2mr^2) - V(r)}} \ , \quad \text{we have} \tag{17}$$

$$\frac{1}{r^2}\frac{dr}{d\theta} = \frac{\sqrt{2m}}{\ell}\sqrt{E - \ell^2u^2/(2m) - \lambda u^{-\alpha}} = -\frac{du}{d\theta} \ , \tag{18}$$

which vanishes when the square root vanishes, i.e., at the solution u_0 of

$$S(u) \equiv (2m/\ell^2)[E - \ell^2 u_0^2/(2m) - \lambda u_0^{-\alpha}] = 0 \ . \tag{19}$$

Now, for a circular orbit, the kinetic energy T is constant and equal to its average:

$$T = \bar{T} = \overline{\frac{r}{2}\frac{dV}{dr}} = \frac{r}{2}\frac{dV}{dr} = \frac{\alpha}{2}\bar{V} = \frac{\alpha}{2}V \ , \quad \text{so that} \tag{20}$$

$$T = E - V(u_0) = \frac{\ell^2}{2m}u_0^2 = \frac{\alpha}{2}V(u_0) = \frac{\alpha}{2}\lambda u_0^{-\alpha} \ \Rightarrow \ u_0^{-(\alpha+2)} = \ell^2/(m\lambda\alpha) \ . \tag{21}$$

Writing $du/d\theta = -\sqrt{S(u)}$, we have

$$\frac{d^2u}{d\theta^2} = -\frac{1}{2\sqrt{S(u)}}S'(u)\frac{du}{d\theta} = \frac{1}{2}S'(u) \ \Rightarrow \ \frac{d^2u}{d\theta^2} = -u + \frac{m\lambda\alpha u^{-(\alpha+1)}}{\ell^2} \ . \tag{22}$$

The right-hand side vanishes at $u = u_0$. Expanding in a Taylor series about $u = u_0$ and defining $\xi = u - u_0$, we find

$$\frac{d^2\xi}{d\theta^2} = -\xi\left[1 + \frac{m\lambda\alpha(\alpha+1)}{\ell^2}u_0^{-(\alpha+2)}\right] \ , \tag{23}$$

which reduces to

$$\frac{d^2\xi}{d\theta^2} = -(\alpha+2)\xi \ . \tag{24}$$

This means that the deviation ξ from a circular orbit exhibits harmonic motion with angular frequency $\sqrt{\alpha+2}$ and hence is periodic with period $2\pi/\sqrt{\alpha+2}$. This is 2π for the Kepler problem $\alpha = -1$ (a non-precessing ellipse); π for the harmonic oscillator $\alpha = 2$, and $2\pi/3$ (a rosette pattern with 3-fold symmetry) for $\alpha = 7$.