ADVANCED CLASSICAL MECHANICS

Physics 316 – Fall Quarter, 2008 – University of Chicago

FINAL EXAMINATION ANSWERS – TUESDAY, DECEMBER 9

This examination contains four problems, worth (10,10,15,5) points, respectively.

(1) (Geometric optics: 10 points): Consider optics problems with axial symmetry, with x the distance from right to left along the optic axis, y the displacement from the optic axis, and φ the angle a ray makes with the optic axis.

(a) (2 points): We defined in class a 2-component vector V consisting of y and φ. We showed that the propagation of a ray from a medium with index of refraction n_i on the right to one with index of refraction n_f on the left through an interface with radius of curvature r was given in the small-angle approximation by

$$V' \equiv \begin{bmatrix} y' \\ \varphi' \end{bmatrix} = T_{\rm lens} \begin{bmatrix} y \\ \varphi \end{bmatrix} \quad , \quad T_{\rm lens} \equiv \begin{bmatrix} 1 & 0 \\ \frac{n_i - n_f}{n_f r} & \frac{n_i}{n_f} \end{bmatrix} \quad . \tag{1}$$

Find a modified version $\tilde{V}$ of the two-component vector such that the corresponding transformation $\tilde{V}' = \tilde{T}_{\rm lens}\tilde{V}$ is a described by a symplectic matrix:

$$(\tilde{T}_{\rm lens})^T J \tilde{T}_{\rm lens} = J \quad , \quad J \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad . \tag{2}$$

Answer: With $T_{\rm lens}$ as defined above, we saw in class that

$$(T_{\rm lens})^T J T_{\rm lens} = (n_i/n_f) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (n_i/n_f) J \quad . \tag{3}$$

If we define the second component of the vector $\tilde{V}$ as $n\varphi$ instead of φ, where n is the index of refraction, the factors of n_i and n_f are absorbed into the corresponding vectors V and V', and one has

$$\tilde{V}' \equiv \begin{bmatrix} y' \\ n'\varphi' \end{bmatrix} = \tilde{T}_{\rm lens} V \quad , \quad \tilde{T}_{\rm lens} \equiv \begin{bmatrix} 1 & 0 \\ \frac{n-n'}{r} & 1 \end{bmatrix} \quad , \quad \tilde{V} \equiv \begin{bmatrix} y \\ n\varphi \end{bmatrix} \quad , \tag{4}$$

(b) (3 points): Fermat's principle says that a light ray takes the path for which the travel time $T = \int (ds/v) = \int (n ds/c)$ is minimum, where $n(x,y)$ is the index of refraction and $ds = (dx^2 + dy^2)^{1/2} = (1 + y'^2)^{1/2} dx$. Treating this as a variational principle in the same sense as the Principle of Least Action, define and interpret the canonical momentum p_y conjugate to y.

Answer: The function $\rho(x,y) =\equiv n(x,y)(1+y'^2)^{1/2}$ satisfies an Euler-Lagrange equation and permits the definition of a momentum p_y conjugate to y:

$$\frac{d}{dx}\frac{\partial \rho}{\partial y'} = \frac{\partial \rho}{\partial y} \quad , \quad p_y \equiv \frac{\partial \rho}{\partial y'} = \frac{n y'}{\sqrt{1 + y'^2}} \quad . \tag{5}$$

The ratio $y'/\sqrt{1+y'^2}$ describes the sine of the angle of the path with respect to the optic axis. Thus in the limit of small angles, p_y is just the quantity $n\varphi$ in part (a).

(c) (3 points): A light ray passes from a medium of index of refraction n_i across a boundary at $x = x_0$ to one with n_f. Show that if n depends only on x, then p_y is conserved, and derive a physical consequence of this fact.

Answer: If $n = n(x)$ is independent of y, then y is a cyclic coordinate, and p_y is conserved. This means that across an interface perpendicular to the optic axis, $n\sin\varphi = n'\sin\varphi'$, which is just Snell's law.

(d) (2 points): From Liouville's theorem, derive a relation between the angular spread of a beam and its dispersion (spatial spread) in y.

Answer: $dp_y dy$ is a conserved element of phase space. For constant n, $dp_y = nd(\sin\varphi)$ and $nd(\sin\varphi)dy = \text{const}$. This says that if we squeeze down on the angular divergence of the beam $d(\sin\varphi)$, we pay the price of a larger spread dy in y.

(2) (10 points): An object is in an orbit about the Sun with eccentricity e and semi-major axis $a = 50$ AU, where 1 AU (Astronomical Unit) is the semi-major axis $a_\oplus$ of the Earth's orbit. Suddenly it undergoes a collision preserving its kinetic energy but pointing it directly toward the Sun. Find the ratio of its kinetic energy T_f when it is 1 AU from the Sun to its initial kinetic energy T_i in the cases of (a) a spherical orbit with $e = 0$, and (b) non-zero eccentricity e when it is knocked from its aphelion or perihelion.

Answer to (a): The initial kinetic energy T_i is the same everywhere for an object in a circular orbit and, by the virial theorem, is equal to minus half the potential energy and minus the total energy:

$$T_i = -\frac{V_i}{2} = +\frac{1}{2}\frac{mGM}{r_i} \ , \quad E_i = -\frac{1}{2}\frac{mGM}{r_i} \ , \tag{6}$$

where m is the mass of the object, G is Newton's gravitation constant, and M is the mass of the Sun. Total energy is conserved, so the total energy at 1 AU is $E_f = E_i = T_f + V_f$, and we know V_f:

$$V_f = -\frac{mGM}{r_f} \quad \Rightarrow \quad T_f = \frac{mGM}{r_f} - \frac{1}{2}\frac{mGM}{r_i} \ . \tag{7}$$

Taking the quotient of T_i and T_f, the common factors of mGM cancel, and

$$\frac{T_f}{T_i} = \left(\frac{1}{r_f} - \frac{1}{2r_i}\right) / \frac{1}{2r_i} = \frac{2r_i - r_f}{r_f} = 99 \ . \tag{8}$$

Answer to (b): For an elliptical orbit with the same semi-major axis $a = 50$ AU, the maximum and minimum radii $r_{\max}$ and $r_{\min}$ satisfy

$$r_{\max} = \frac{1}{C(1-e)} \ , \quad r_{\min} = \frac{1}{C(1+e)} \ , \quad C \equiv \frac{m\lambda}{\ell^2} \ , \quad \lambda \equiv mGM \ , \tag{9}$$

where ℓ is the orbital angular momentum. The total energy E and semi-major axis are related by $E = -\lambda/(2a)$. The initial potential energy at aphelion is

$$V_{\rm ap} = -\frac{\lambda}{r_{\rm max}} \quad \Rightarrow \quad T_{\rm ap} = E + \frac{\lambda}{r_{\rm max}} = -\frac{\lambda}{2a} + \frac{\lambda}{r_{\rm max}} \quad , \tag{10}$$

while the final potential energy, by assumption, is

$$V_f = -\frac{\lambda}{a_\oplus} \quad \Rightarrow \quad T_f = E - V_f = -\frac{\lambda}{2a} + \frac{\lambda}{a_\oplus} \quad . \tag{11}$$

Then for an object starting at aphelion,

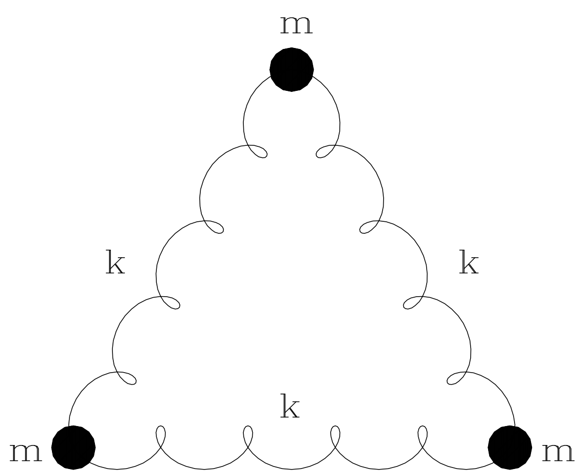
$$\frac{T_f}{T_i} = \left(\frac{1}{a_\oplus} - \frac{1}{2a}\right) / \left(\frac{1}{r_{\rm max}} - \frac{1}{2a}\right) = \frac{2a - a_\oplus}{2a - r_{\rm max}} \frac{r_{\rm max}}{a_\oplus} = \frac{2a - a_\oplus}{a_\oplus} \frac{r_{\rm max}}{r_{\rm min}} = \frac{2a - a_\oplus}{a_\oplus} \frac{1+e}{1-e} \, . \tag{12}$$

A similar calculation holds with $e \to -e$ for an object knocked from perihelion.

(3) (10 points): Three equal masses m are connected by springs with equal constants k such that their equilibrium position is an equilateral triangle with each side equal to a. The (x, y) coordinates of the equilibrium positions may be taken to be

$$\vec{x}_1^0 = (-\frac{a}{2}, 0) \quad , \quad \vec{x}_2^0 = (\frac{a}{2}, 0) \quad , \quad \vec{x}_3^0 = (0, \frac{\sqrt{3}}{2} a) \quad . \tag{13}$$

The equilibrium configuration is illustrated below.



Let x_i $(i = 1, 2, 3)$ denote the *displacements* of these coordinates from their equilibrium positions. (a) Show that the introduction of Jacobi coordinates

$$\vec{\rho} \equiv \frac{\vec{x}_2 - \vec{x}_1}{\sqrt{2}} \quad , \quad \vec{\eta} \equiv \frac{2\vec{x}_3 - \vec{x}_1 - \vec{x}_2}{\sqrt{6}} \tag{14}$$

leads to a Hamiltonian in which the center-of-mass, described by $\vec{X} \equiv (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)/3$, undergoes free motion; the kinetic energy is a function of the squared sum $\dot{\vec{\rho}}^2 + \dot{\vec{\eta}}^2$; and the potential may be written as a 4×4 symmetric matrix in the basis $[\rho_x, \rho_y, \eta_x, \eta_y]$ as

$$V = \frac{3}{8}k \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix} \quad . \tag{15}$$

Answer: Substituting in the expression for the kinetic energy, we find

$$T = \frac{m}{2}(\dot{\vec{x}}_1^2 + \dot{\vec{x}}_2^2 + \dot{\vec{x}}_3^2) = \frac{3m}{2}\dot{\vec{X}}^2 + \frac{m}{2}(\dot{\vec{\rho}}^2 + \dot{\vec{\eta}}^2) \ , \tag{16}$$

while for the potential energy describing deviations from equilibrium positions we have the pair contributions

$$V_{12} = \frac{k}{2}(x_2 - x_1)^2 \ , \ V_{13} = \frac{k}{8}(x_3 + \sqrt{3}y_3 - x_1 - \sqrt{3}y_1)^2 \ , \ V_{23} = \frac{k}{8}(-x_3 + \sqrt{3}y_3 + x_2 - \sqrt{3}y_2)^2 \ . \tag{17}$$

Re-expressing the sum of these in terms of the Jacobi coordinates, we find $V = V_{12} + V_{13} + V_{23} = V_{ij}q_iq_j$, where $\vec{q}$ is a four-component vector with components $\rho_x, \rho_y, \eta_x, \eta_y$ and V_{ij} is given above. V is independent of X; the kinetic energy term $(3m/2)\dot{X}^2$ corresponds to free motion of the system as a whole.

(b) Find the eigenvectors and eigenvalues of V (it should be easy to spot them by inspection) and interpret them physically.

Answer: The kinetic energy is diagonal in the $\rho_x, \rho_y, \eta_x, \eta_y$ basis, so it is sufficient to diagonalize the potential energy to find the eigenmodes. The eigenvectors of $8V/3k$ separate into two of the form $[a, 0, 0, b]$ and two others of the form $[0, c, d, 0]$. The largest eigenvalue of $8V/3k$ is 4, corresponding to the eigenvector $[1, 0, 0, 1]$, while an eigenvalue of 2 corresponds to $[1, 0, 0, -1]$. An eigenvalue of 2 corresponds to $[0, 1, 1, 0]$, while a zero eigenvalue corresponds to the rotational mode $[0, 1, -1, 0]$. The two modes with eigenvalue 2 are degenerate. In the mode with eigenvector $[1, 0, 0, -1]$, when mass 3 moves toward the center of mass of masses 1 and 2, those masses move apart from one another. The other eigenvector with eigenvalue 2 is a linear combination of the same motion with mass 1 moving against 2 and 3, and 2 moving against 3 and 1. Finally, the eigenvalue 4 and eigenvector $[1, 0, 0, 1]$ correspond to a uniform stretching motion.

(4) (5 points): Consider a uniform right cylinder with radius a and height h. (a) Write and solve the Euler equations for the gravity-free motion of the cylinder. (b) For what ratio of h/a does the motion approach that of a symmetric top? What happens to your solution in (a) as this limit is approached?

Answer: (a) Two moments of intertia are equal: $I_1 = I_2$, while the third (I_3, the moment about the cylinder axis), is in general different. Then the Euler equations reduce to $I_1\dot{\omega}_1 = \omega_2\omega_3(I_1 - I_3)$, $I_1\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1)$, $I_3\dot{\omega}_3 = 0$, so $\dot{\omega}_3 = 0$ and ω_3 is constant. Thus ω_1 and ω_3 undergo simple harmonic motion, 90° out of phase, with angular frequency $\omega_3(I_1 - I_3)/I_1$. (b) Short calculations give $I_3 = Ma^2/2$, $I_1 + I_2 = M[(a^2/2) + (h^2/6)]$ so $I_1 = I_2 = M[(a^2/4) + (h^2/12)]$. This is equal to I_3 when $h = \sqrt{3}a$. The oscillation frequency $\omega_3(I_1 - I_3)/I_1$ goes to zero in this limit.