Homework 9 - Solutions

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Problem 1

Suppose you had two particles of spin 2, each in a state with $S_z = 0$. If you measured the *total* angular momentum of this system, given that the *orbital* angular momentum is zero, what values might you get, and what is the probability of each? Check that they add up to 1.

Solution

If there is no orbital angular momentum, then $L = 0$ and the total angular momentum $J = L + S = S$. Therefore, $J = 2$ for each state and $J_z = 0$. To find the total angular momentum of the system, one needs to find the product of two $|J\,J_z\rangle = |2\,0\rangle$ states. Using the Clebsch-Gordon coefficients page, we find

$$|2\,0\rangle\,|2\,0\rangle \quad = \quad \sqrt{18/35}\,|4\,0\rangle + 0\,|3\,0\rangle - \sqrt{2/7}\,|2\,0\rangle + 0\,|1\,0\rangle + \sqrt{1/5}\,|0\,0\rangle\,.$$

The probability to be in any particular state is given by the square of the coefficient in front of the state. Therefore, the state can have $J = 4$ and $J_z = 0$ with a probability of 18/35, $J = 2$ and $J_z = 0$ with a probability of 2/7, and $J = 0$ and $J_z = 0$ with a probability of 1/5. Adding $18/35 + 2/7 + 1/5$, we do in fact find a total probability of 1.

Problem 2

Suppose an electron is in the state $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

(a) If you measured S_x, what values might you get, and what is the probability of each?
(b) If you measured S_y, what values might you get, and what is the probability of each?
(c) If you measured S_z, what values might you get, and what is the probability of each?

Solution

(a) Let's call the electron spinor $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, such that $\alpha = 1/\sqrt{5}$ and $\beta = 2/\sqrt{5}$. Now, the matrix corresponding to S_x is

$$\hat{S}_x \quad = \quad \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

This has eigenvalues of $\pm\hbar/2$ and respective normalized eigenvectors of

$$\chi_\pm \quad = \quad \begin{pmatrix} 1/\sqrt{2} \\ \pm 1/\sqrt{2} \end{pmatrix}.$$

Our spinor can be written as a linear combination of these eigenvectors

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad = \quad a\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + b\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}. \tag{2}$$

Now, the probability of $S_x = \hbar/2$ is $|a|^2$ and the probability of $S_x = -\hbar/2$ is $|b|^2$. We find from Eq. 2 that $a = 1/\sqrt{2}\,(\alpha + \beta)$ and $b = 1/\sqrt{2}\,(\alpha - \beta)$. Therefore, the probability of $S_x = \hbar/2$ is $a^2 = 9/10$ and of $S_x = -\hbar/2$ is $b^2 = 1/10$.

(b) The matrix corresponding to S_y is

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

This has eigenvalues of $\pm\hbar/2$ and respective normalized eigenvectors of

$$\chi_\pm = \begin{pmatrix} 1/\sqrt{2} \\ \pm i/\sqrt{2} \end{pmatrix}.$$

Again, we can decompose the spinor as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} + b\begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}. \tag{4}$$

This gives $a = 1/\sqrt{2}\,(\alpha - i\beta)$ and $b = 1/\sqrt{2}\,(\alpha + i\beta)$. Therefore, taking the absolute squares of these gives us that the probability of $S_y = \hbar/2$ is $a^2 = 1/2$ and of $S_y = -\hbar/2$ is $b^2 = 1/2$.
(c) The matrix corresponding to S_z is

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

This has eigenvalues of $\pm\hbar/2$ and respective normalized eigenvectors of

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The spinor decomposes like

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6}$$

This gives $a = \alpha$ and $b = \beta$ giving the probability of $S_z = \hbar/2$ is $a^2 = 1/5$ and of $S_z = -\hbar/2$ is $b^2 = 4/5$.

Problem 3

Use the results of Problem 4.19 in Griffiths to show that
(a) The commutator, $[A, B] \equiv AB - BA$, of two Pauli matrices is $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$.
(b) The *anticommutator*, $\{A, B\} \equiv AB + BA$, is $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$.
(c) For any two vectors **a** and **b**, $(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b})$.

Solution

(a) Since $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, we see that

$$\begin{aligned} [\sigma_i, \sigma_j] &= \sigma_i\sigma_j - \sigma_j\sigma_i \\ &= \delta_{ij} + i\epsilon_{ijk}\sigma_k - \delta_{ji} - i\epsilon_{jik}\sigma_k. \end{aligned}$$

Now, since $\delta_{ij} = \delta_{ji}$ and $\epsilon_{ijk} = -\epsilon_{jik}$ by definition, the delta functions cancel and the Levi-Civita tensors add to give

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

(b) Using the same relation, we find

$$\begin{aligned}\{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_j\sigma_i \\ &= \delta_{ij} + i\epsilon_{ijk}\sigma_k + \delta_{ji} + i\epsilon_{jik}\sigma_k \\ &= 2\delta_{ij},\end{aligned}$$

by the same reasoning as in part (a).
(c) The dot product of two vectors is given by (with summation over repeated indices implied)

$$\sigma \cdot \mathbf{a} = \sigma_i a_i.$$

Therefore, we see

$$\begin{aligned}(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) &= \sigma_i a_i \sigma_j b_j \\ &= \sigma_i \sigma_j a_i b_j.\end{aligned} \tag{7}$$

Now, again $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, so Eq. 7 becomes

$$\begin{aligned}(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) &= (\delta_{ij} + i\epsilon_{ijk}\sigma_k) a_i b_j \\ &= a_j b_j + i\sigma_k \epsilon_{ijk} a_i b_j \\ &= \mathbf{a} \cdot \mathbf{b} + i\sigma_k (\mathbf{a} \times \mathbf{b})_k \\ &= \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b}),\end{aligned}$$

where we have used the fact that $(\mathbf{a} \times \mathbf{b})_k = \epsilon_{ijk} a_i b_j$.

Problem 4

Determine the isospin assignments $|I\, I_3\rangle$ for each of the following particles (refer to the Eightfold Way diagrams in Chapter 1 of Griffiths): Ω^-, Σ^+, Ξ^0, ρ^+, η, $\bar{K}^0$.

Solution

On the Eightfold Way diagrams, the particles are grouped together along similar horizontal lines to form isospin multiplets. For instance, in the baryon octet on page 33, the neutron and proton are both on the same horizontal line. Therefore, they form an isospin doublet of total isospin $I = 1/2$ (singlets have $I = 0$, doublets $I = 1/2$, triplets $I = 1$, and so forth). However, it is important to note that the second horizontal line on the baryon octet has four particles, but these form a triplet and a singlet, not a quadruplet. This happens when particles share a similar position on the diagram. Therefore, the Σ's form an isospin triplet with $I = 1$ and the Λ is a singlet with $I = 0$. Finally, higher I_3 states occur the further right you go on the diagram. Therefore, Σ^+ has $|I\, I_3\rangle = |1\, 1\rangle$, Σ^0 has $|I\, I_3\rangle = |1\, 0\rangle$, and Σ^- has $|I\, I_3\rangle = |1\, -1\rangle$.

Now, looking at the baryon decuplet, we see that Ω^- forms a singlet and therefore has a state $|I\, I_3\rangle = |0\, 0\rangle$. We already saw that Σ^+ is $|I\, I_3\rangle = |1\, 1\rangle$. In the baryon octet, we see that Ξ^0 is in a doublet and has isospin state $|I\, I_3\rangle = \left|\frac{1}{2}\, \frac{1}{2}\right\rangle$. For ρ^+, it's a little different because this isn't in one of the meson multiplets. However, we do know that there are three possible charge states for the ρ and therefore this should form a triplet (much like the π's). In this case, ρ^+ has an isospin state of $|I\, I_3\rangle = |1\, 1\rangle$. In the meson octet, we see that the η is the singlet state sharing a position with π^0, so it has a state $|I\, I_3\rangle = |0\, 0\rangle$. Finally, looking at the meson octet, we see that $\bar{K}^0$ state is $|I\, I_3\rangle = \left|\frac{1}{2}\, \frac{1}{2}\right\rangle$.

Problem 5

What are the possible total isospins for the following reactions: (a) $K^- + p \to \Sigma^- + \pi^+$; (b) $K^- + p \to \Sigma^+ + \pi^-$. Find the ratio of the two cross sections, assuming one or the other isospin channel dominates.

Solution

(a) Let's look at the reaction $K^- + p \to \Sigma^- + \pi^+$. The isospin state of the left hand side is (using the table of Clebsch-Gordon coefficients to decompose the isospin product)

$$K^- + p: \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle \left|\frac{1}{2}\ \frac{1}{2}\right\rangle \;=\; \frac{1}{\sqrt{2}}\left|1\ 0\right\rangle - \frac{1}{\sqrt{2}}\left|0\ 0\right\rangle. \tag{8}$$

The isospin state of the right hand side is

$$\Sigma^- + \pi^+ : \left|1\ -1\right\rangle \left|1\ 1\right\rangle \;=\; \frac{1}{\sqrt{6}}\left|2\ 0\right\rangle - \frac{1}{\sqrt{2}}\left|1\ 0\right\rangle + \frac{1}{\sqrt{3}}\left|0\ 0\right\rangle. \tag{9}$$

Since the initial isospin is either $I = 1$ or $I = 0$, the $I = 2$ ket in the final state won't actually occur in order to conserve isospin. Therefore, the isospin is either $I = 1$ or $I = 0$. The scattering amplitude for this reaction is then (given that there are two amplitudes, $\mathcal{M}_1$ for the $I = 1$ state and $\mathcal{M}_0$ for the $I = 0$ state)

$$\begin{aligned} \mathcal{M}_a &= \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\mathcal{M}_1 + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{3}}\mathcal{M}_0 \\ &= -\frac{1}{2}\mathcal{M}_1 - \frac{1}{\sqrt{6}}\mathcal{M}_0. \end{aligned} \tag{10}$$

(b) Now, for the reaction $K^- + p \to \Sigma^+ + \pi^-$, the isospin state for the right hand side is

$$\Sigma^+ + \pi^- : \left|1\ 1\right\rangle \left|1\ -1\right\rangle \;=\; \frac{1}{\sqrt{6}}\left|2\ 0\right\rangle + \frac{1}{\sqrt{2}}\left|1\ 0\right\rangle + \frac{1}{\sqrt{3}}\left|0\ 0\right\rangle. \tag{11}$$

Again, we have either $I = 1$ or $I = 0$ for the isospin of the reaction. For this case, the scattering amplitude is

$$\begin{aligned} \mathcal{M}_b &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\mathcal{M}_1 + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{3}}\mathcal{M}_0 \\ &= \frac{1}{2}\mathcal{M}_1 - \frac{1}{\sqrt{6}}\mathcal{M}_0. \end{aligned} \tag{12}$$

Now, the ratio of the cross sections of these interactions goes as the ratio of the squares of the amplitudes, that is

$$\begin{aligned} \frac{\sigma_a}{\sigma_b} &= \frac{\left|\mathcal{M}_a\right|^2}{\left|\mathcal{M}_b\right|^2} \\ &= \frac{\left|\frac{1}{2}\mathcal{M}_1 + \frac{1}{\sqrt{6}}\mathcal{M}_0\right|^2}{\left|\frac{1}{2}\mathcal{M}_1 - \frac{1}{\sqrt{6}}\mathcal{M}_0\right|^2} \\ &= \frac{\frac{1}{4}\left|\mathcal{M}_1\right|^2 + \frac{1}{\sqrt{6}}\left|\mathcal{M}_1\right|\left|\mathcal{M}_0\right|\cos\theta + \frac{1}{6}\left|\mathcal{M}_0\right|^2}{\frac{1}{4}\left|\mathcal{M}_1\right|^2 - \frac{1}{\sqrt{6}}\left|\mathcal{M}_1\right|\left|\mathcal{M}_0\right|\cos\theta + \frac{1}{6}\left|\mathcal{M}_0\right|^2}, \end{aligned} \tag{13}$$

where θ is the angular separation in the complex plane of the two complex numbers $\mathcal{M}_1$ and $\mathcal{M}_0$. Now, if $\mathcal{M}_1 \gg \mathcal{M}_0$, we can divide through by $\frac{1}{4}\left|\mathcal{M}_1\right|^2$ and ignore terms of order $\left|\mathcal{M}_0\right|^2 / \left|\mathcal{M}_1\right|^2$ to see

$$\begin{aligned} \frac{\sigma_a}{\sigma_b} &\simeq \frac{1 + \frac{4}{\sqrt{6}}\frac{\left|\mathcal{M}_0\right|}{\left|\mathcal{M}_1\right|}\cos\theta}{1 - \frac{4}{\sqrt{6}}\frac{\left|\mathcal{M}_0\right|}{\left|\mathcal{M}_1\right|}\cos\theta} \\ &= \left(1 + \frac{4}{\sqrt{6}}\frac{\left|\mathcal{M}_0\right|}{\left|\mathcal{M}_1\right|}\cos\theta\right)\left(1 - \frac{4}{\sqrt{6}}\frac{\left|\mathcal{M}_0\right|}{\left|\mathcal{M}_1\right|}\cos\theta\right)^{-1} \\ &\simeq 1 + \frac{8}{\sqrt{6}}\frac{\left|\mathcal{M}_0\right|}{\left|\mathcal{M}_1\right|}\cos\theta, \end{aligned} \tag{14}$$

where we get the last term by binomial expansion. In the same way, if $\mathcal{M}_0 \gg \mathcal{M}_1$ we can do the same to get

$$\begin{aligned}\frac{\sigma_a}{\sigma_b} &\simeq \frac{1+\sqrt{6}\frac{|\mathcal{M}_1|}{|\mathcal{M}_0|}\cos\theta}{1-\sqrt{6}\frac{|\mathcal{M}_1|}{|\mathcal{M}_0|}\cos\theta} \\ &\simeq 1+2\sqrt{6}\frac{|\mathcal{M}_1|}{|\mathcal{M}_0|}\cos\theta. \end{aligned} \tag{15}$$

Therefore, from Eqs. 14 and 15 we see that the reaction $K^- + p \to \Sigma^- + \pi^+$ is slightly more likely than the reaction $K^- + p \to \Sigma^+ + \pi^-$. This slight discrepancy is a bit more pronounced in the case that $\mathcal{M}_0 \gg \mathcal{M}_1$. However, they're still very similar cross sectional ratios.

Problem 6

(a) The α particle is a bound state of two protons and two neutrons, that is, a ^{4}He nucleus. There is no isotope of hydrogen with an atomic weight of four (^{4}H), nor of lithium ^{4}Li. What do you conclude about the isospin of an α particle?
(b) The reaction $d + d \to \alpha + \pi^0$ has never been observed. Explain why not.
(c) Would you expect ^{4}Be to exist? How about a bound state of four neutrons?

Solution

(a) If ^{4}Li and ^{4}H existed, we would expect that these would perhaps form an isospin triplet (with $I = 1$) with ^{4}He, with ^{4}Li having $I_3 = +1$, ^{4}He having $I_3 = 0$, and ^{4}H having $I_3 = -1$. However, since these other two states don't exist, ^{4}He should instead just be part of a singlet and therefore have $I = 0$.
(b) Now, deuterons are also singlets, and therefore the left hand side of the reaction has isospin $I = 0$. On the other hand, since pions are in a triplet state, the right hand side has an isospin of $I = 1$. Therefore, this reaction would violate isospin and therefore won't occur via the strong interaction, which is the only force capable of making this possible.
(c) We would not expect ^{4}Be or a bound state of four neutrons to exist because these would have $I_3 = +2$ and $I_3 = -2$, respectively, and thus be part of an $I = 2$ quintuplet. The $I_3 = +1$ and $I_3 = -1$ states of this quintuplet would be ^{4}Li and ^{4}H, respectively, which we have already found don't exist. Therefore, since we can't seem to find other members of a possible quintuplet, we don't expect to see ^{4}Be or a bound state of four neutrons, either.