| 1 | 2 | 3 | 4 | | | | | | | /Total |
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NAME: Solutions.

(Last) (First) (Initial)

**P142 – Midterm Exam, Feb. 6, 2004**

**Show all work.** Make sure that your answers include the proper units. For vector quantities, be sure to include both magnitude and direction.

Vector derivatives:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

1. (a) State Gauss's law in both integral and differential forms.

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{encl} = 4\pi \int \rho \, dV$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

(b) A spherical charge distribution is described by the following charge density:

$\rho(r) = \rho_0 \frac{r}{a}$ when $r \leq a$ and $\rho = 0$ when $r > a$.

Use Gauss's Law to calculate the electric field (magnitude and direction) as a function of r.

For $r < a$:

$$\int \vec{E} \cdot d\vec{a} = 4\pi \int^{r} \rho \, dV$$

$$E \, 4\pi r^2 = 4\pi \int_0 \rho_0 \frac{r}{a} 4\pi r^2 dr$$

$$\vec{E} = \frac{\rho_0 \pi r^2}{a} \hat{r} \quad \text{(radially outward)}$$

For $r > a$:

$$E \cdot 4\pi r^2 = 4\pi \int_0^a \rho_0 \frac{r}{a} 4\pi r^2 dr$$

$$= 4\pi \rho_0 \frac{a^4}{4a} 4\pi$$

$$\vec{E} = \frac{\rho_0 \pi a^3}{r^2} \hat{r}$$

(c) Calculate the electric potential φ(r) for all r. (Set φ=0 at r=∞.)

$$\varphi(r) = -\int_\infty^r \vec{E} \cdot d\vec{s} = -\int_\infty^r E \, dr'$$

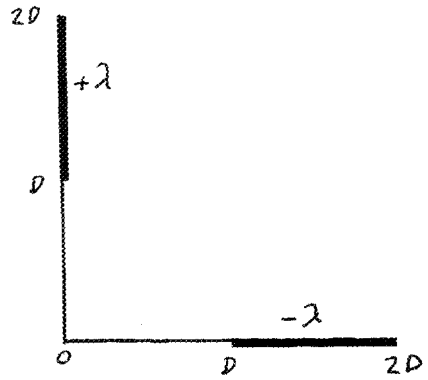
For $r > a$, $\varphi(r) = -\int \frac{\rho_0 \pi a^3}{r^2} dr = \frac{\rho_0 \pi a^3}{r}$

For $r < 0$,

$$\varphi(r) = \frac{\rho_0 \pi a^3}{a} - \int_a^r \frac{\rho_0 \pi r^2}{a} dr = \rho_0 \pi a^2 - \left[\frac{\rho_0 \pi r^3}{3a}\right]_a^r$$

$$= \rho_0 \pi a^2 - \frac{\rho_0 \pi r^3}{3a} + \frac{\rho_0 \pi a^3}{3a} = \frac{4}{3}\rho_0 \pi a^2 - \frac{\rho_0 \pi r^3}{3a}$$

2. A thin rod with uniform charge density $-\lambda$ (esu/cm) extends from x=D to x=2D along the x-axis. A similar rod with uniform charge density $+\lambda$ extends from y=D to y=2D along the y axis.




(a) Find the work required to move a positive point charge (+q) from infinity to the origin. Does the result depend on the path you choose? Explain why or why not.

Defining $\varphi(\infty) = 0$, $\varphi(\text{origin}) = 0$. (For every positive charge element that is a certain distance from the origin, there's a corresponding negative charge element.

Work from $\infty$ to origin $= +q\,\Delta\varphi = 0$.

The result is independent of path because the electrostatic force is conservative.

(b) Calculate the force (magnitude and direction) that is required to keep the point charge at the origin.

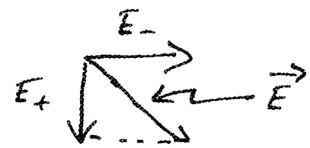
From the negative charge distribution:

$$\vec{E}_- = \hat{\imath}\int_D^{2D} \frac{\lambda}{x^2}\,dx = \hat{\imath}\left(\frac{\lambda}{D} - \frac{\lambda}{2D}\right) = \hat{\imath}\,\frac{\lambda}{2D}$$

For the positive charge distribution,

$$\vec{E}_+ = -\hat{\jmath}\int_D^{2D} \frac{\lambda}{y^2}\,dy = -\hat{\jmath}\,\frac{\lambda}{2D}$$

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\lambda}{2D}(\hat{\imath} - \hat{\jmath})$$

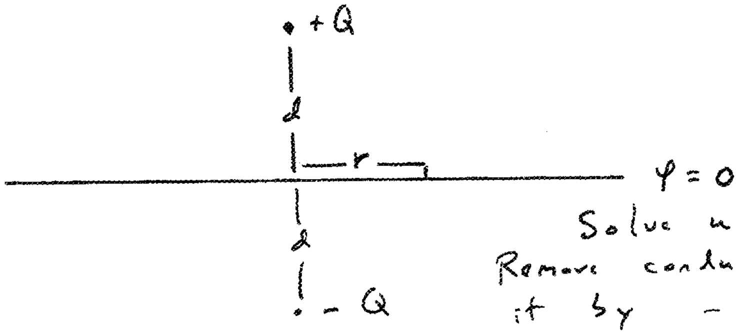



Force to keep particle at origin is $\vec{F} = -q\vec{E}$

$$\vec{F} = -q\frac{\lambda}{2D}(\hat{\imath} - \hat{\jmath}) = \frac{q\lambda}{2D}(\hat{\jmath} - \hat{\imath})$$

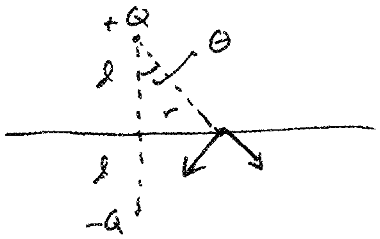
3. Suppose a point charge +Q is suspended a distance d above an infinite conducting plane.

(a) Find the electric field (magnitude and direction) at the surface of the conductor, $\vec{E}(r)$, where r is the distance on the surface of the plane measured from the perpendicular line from the plane to the charge.




Solve using method of images.
Remove conducting plane and replace it by $-Q$ a distance $d$ below plane.

At surface of plane, $\vec{E} = \vec{E}_{+Q} + \vec{E}_{-Q}$:




Component in x direction cancel; y components are both in $-y$ direction and add.

$$\vec{E} = -2\frac{Q}{R^2+d^2}\cos\theta\,\hat{j}$$

$$= -\frac{2Q}{r^2+d^2}\frac{d}{[r^2+d^2]^{1/2}}\hat{j}$$

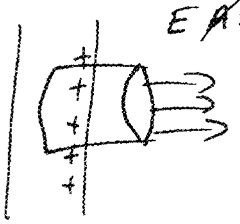
$$= -\frac{2Qd}{[r^2+d^2]^{3/2}}\hat{j}$$

(As expected, $\vec{E}$ is $\perp$ surface.)

(b) What is the surface charge density, $\sigma(r)$, induced on the surface of the conducting plane?

From Gauss's Law, $|E_{SURF}| = 4\pi\sigma$.

$E\cancel{A} = 4\pi\sigma\cancel{A}$

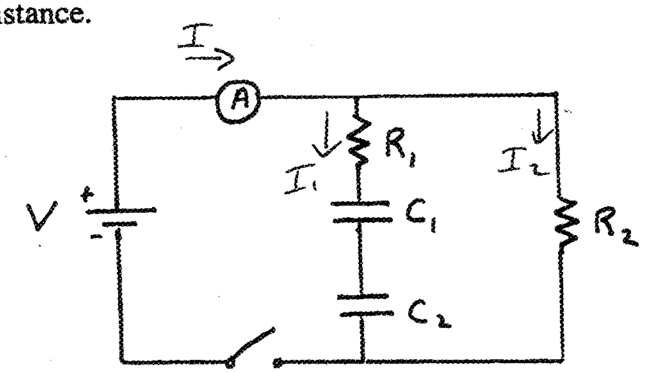


Therefore, $\sigma = \dfrac{E}{4\pi}$

$$= -\frac{1}{4\pi}\frac{2Qd}{[r^2+d^2]^{3/2}}$$

(negative surface charge density)

4. Consider the circuit drawn below. The ammeter (labeled A) measures the current flowing through the battery. Initially, the switch is open and the capacitors are uncharged. Remember that an ammeter has negligible resistance.




(a) What is the current measured by the ammeter just after the switch is closed at t=0?

At $t=0$, $\Delta V$ across $C_1, C_2 = 0$.

$\Rightarrow V = I_1 R_1$, $V = I_2 R_2$ (resistors in ||)

$I = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

(b) What is the current measured by the ammeter a long time after the switch is closed (once the circuit has reached a steady state)?

After a long time, $I_1 = 0$.

$V = I_2 R_2 = I R_2$

$\Rightarrow I = \frac{V}{R_2}$

(c) A long time after the switch is closed, when equilibrium is reached, what is the voltage across each capacitor and each resistor?

As discussed in class, charges on $C_1$ and $C_2$ will be equal:

$C_1$ $C_2$ $+Q \,|\,|\, -Q \quad +Q \,|\,|\, -Q$. $V = \frac{Q}{C_1} + \frac{Q}{C_2}$ ; $I_1 = 0$

$V_{C_1} = V/\left(1 + \frac{C_1}{C_2}\right)$

$R_2 = V$

$V_{C_2} = V/\left(1 + \frac{C_2}{C_1}\right)$

$R_1 = 0$