Physics 142 Homework 8 Solutions

Q1) $\vec{E} = E_0 \sin\left(\frac{2\pi}{\lambda}(z+ct)\right)(\hat{i}+\hat{j})$

$$\Rightarrow \vec{\nabla}\times\vec{E} = E_0 \,\mathrm{Det}\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin\left(\frac{2\pi}{\lambda}(z+ct)\right) & \sin\left(\frac{2\pi}{\lambda}(z+ct)\right) & 0 \end{pmatrix}$$

$$= -\hat{i}\, E_0 \frac{2\pi}{\lambda}\cos\left(\frac{2\pi}{\lambda}(z+ct)\right) + \hat{j}\, E_0 \frac{2\pi}{\lambda}\cos\left(\frac{2\pi}{\lambda}(z+ct)\right)$$

$$= -E_0 \frac{2\pi}{\lambda}\cos\left(\frac{2\pi}{\lambda}(z+ct)\right)(\hat{i}-\hat{j})$$

$$\Rightarrow \frac{1}{c}\frac{d\vec{B}}{dt} = E_0 \frac{2\pi}{\lambda}\cos\left(\frac{2\pi}{\lambda}(z+ct)\right)(\hat{i}-\hat{j}) \quad \{\text{using Maxwells' laws}\}$$

$$\Rightarrow \vec{B} = E_0 \sin\left(\frac{2\pi}{\lambda}(z+ct)\right)(\hat{i}-\hat{j})$$

Q2) We know we are in free space. So $J = 0 = \rho$.

Now maxwells' laws give us

$$\frac{1}{c}\frac{\partial E}{\partial t} = \vec{\nabla}\times B \quad \& \quad \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\vec{\nabla}\times\vec{\nabla}\times\vec{E}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \nabla^2\vec{E}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \nabla^2\vec{E} \quad \{\text{as } \rho = 0 \text{ by gauss law } \vec{\nabla}\cdot\vec{E} = 0\}$$

$$\vec{E} = E_0 \sin(kx+\omega t)\,\hat{j}$$

$\Rightarrow -\frac{\omega^2}{c^2}E_0\sin(kx+\omega t) = -k^2 E_0 \sin(kx+\omega t)$ when we put $\vec{E}$ into the wave equation. $\Rightarrow \frac{\omega}{c} = \pm k$

b) $\omega = 10^{10} s^{-1}$ $\quad \therefore \lambda = \frac{2\pi c}{\omega} = \frac{2\pi \times 10^{10} \times 3}{10^{11}} = 6\pi$ cm

The energy density $= \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2) = \frac{1}{8\pi} 2E_0^2 \sin^2(kx+\omega t) = \frac{E_0^2}{4\pi}\sin^2(kx+\omega t)$

So the mean energy density $= \frac{E_0^2}{8\pi} = \frac{(0.05)^2}{8\pi} \approx 10^{-4}$ ergs/cm³

Similarly the mean power density $= \frac{E_0^2 c}{4\pi}$ $\quad \frac{(0.05)^2 \times 3\times 10^{10}}{4\pi} \approx 6\times 10^6$ ergs/cm s.

Q3) We have in SI & $\rho = 0 = \vec{J}$

$$\vec{\nabla}\cdot\vec{E} = 0 \;;\; \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad — \quad (1)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0\vec{J}$$

$$= \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{as } \vec{J} = 0 \qquad — (2)$$

⇒ Now (1) using $\vec{E} = E_0 \sin(y - vt)\hat{k}$, $\vec{B} = B_0 \sin(y - vt)\hat{i}$ implies

$$E_0 \cos(y - vt)\hat{i} = B_0 v \cos(y - vt)\hat{i}$$

$$\Rightarrow E_0 = B_0 v$$

Also using (1) we have $\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\vec{\nabla}\times\vec{\nabla}\times\vec{E}$

$$\Rightarrow \mu_0\varepsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} = \vec{\nabla}^2\vec{E}$$

So we have $\cancel{-}\mu_0\varepsilon_0 v^2 \cancel{E_0 \sin(y-vt)} = \cancel{-E_0\sin(y-vt)}$

$$\Rightarrow v^2 = \frac{1}{\mu_0\varepsilon_0} \quad \text{or } v = \pm\frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad \Rightarrow B_0 = \sqrt{\mu_0\varepsilon_0}\,E_0$$

Q4) We have $f = 10^8\ Hz$

$$\Rightarrow \omega = 2\pi f = 2\times10^8\pi \text{ rad/sec.}$$

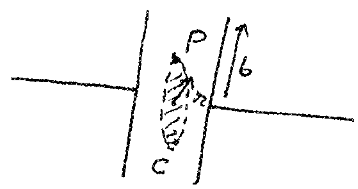
$$\Rightarrow k = \frac{c}{2\pi\times10^8} = \frac{3\times10^{10}}{2\pi\times10^8}\ cm^{-1} \approx 0.47\times10^2 = 47\ cm^{-1}$$

As the direction of propagation is $-\hat{i}$ & $\vec{E}$ is perpendicular to $\hat{z}$ we need that $\vec{E}$ point in the y direction, so $\vec{B}$ must point is $\hat{z}$ direction.

So if $\vec{E} = E_0 \sin(47x + 6.28\times10^8 t)\hat{j}$ $\vec{B} = -\vec{E}_0 \sin(47x + 6.28\times10^8 t)\hat{k}$.

Q5) In the capacitor $\vec{J} = 0$ so we have

$$\int_c \vec{B}\cdot d\vec{s} = \frac{1}{c}\int_s \frac{\partial\vec{E}}{\partial t}\cdot d\vec{A}$$



Now $\vec{E} = 4\pi\sigma\,\hat{i}$ $\Rightarrow \frac{\partial\vec{E}}{\partial t} = 4\pi\frac{\partial\sigma}{\partial t}\hat{i} = -\frac{4\pi}{A}\frac{\partial Q}{\partial t} = \frac{4\pi I}{A}$ which constant over the whole disc of radius r, and $\frac{\partial Q}{\partial t} = -I$ as I carays current away from the capacitor.

$$\Rightarrow B\cdot 2\pi r = \frac{1}{c}\frac{\cancel{4}^{2}\pi I}{A}\pi r^2 = \frac{1}{c}\cancel{4\pi}\frac{2Ir}{b^2} = \frac{2Ir}{cb^2}.$$

Q6) We have the empty capacitor is C_0. So if we half its width the capacitance becomes. $2C_0$. Now if there half capacitors is filled with dielectric of dielectric constant ϵ the capacitance $\frac{2C_0}{\epsilon}$.

Now the first arrangement looks like two capacitors in series $\Rightarrow \frac{1}{C_{eq}} = \frac{\epsilon}{2C_0} + \frac{1}{2C_0}$

$\Rightarrow C_{eq} = \frac{2C_0}{(1+\epsilon)}$. Similarly for second arrangement the area is halved so the capacitance of each piece becomes $C_0/2$. & $C_0/2\epsilon$ fore each piece respectively

$\therefore C_{eq} = \frac{C_0}{2}\left(\frac{1}{\varepsilon}+1\right) = \frac{C_0(1+\varepsilon)}{2\varepsilon}$ as the two capacitors are in parallel.