Physics 142: Homework 3

January 28, 2004

Q1)

We have the total energy stored for a particular charge configuration ρ and potential ϕ is

$$U = \frac{1}{2}\int \rho\phi \, dv \tag{1}$$

and for an electric $\mathbf{E}$ the energy stored is

$$U = \frac{1}{8\pi}\int \mathbf{E}\cdot\mathbf{E}\, dv. \tag{2}$$

Starting with 2 we have

$$\begin{aligned}
U &= \frac{1}{8\pi}\int \mathbf{E}\cdot\mathbf{E}\, dv && (3)\\
&= \frac{1}{8\pi}\int (\nabla\phi)\cdot(\nabla\phi)\, dv && (4)\\
&= \frac{1}{8\pi}\int \nabla\cdot(\phi\nabla\phi)\, dv - \frac{1}{8\pi}\int \phi\nabla^2\phi\, dv && (5)\\
&= \frac{1}{8\pi}\int (\phi\nabla\phi)\cdot\ d\mathbf{A} + \frac{1}{2}\int \rho\phi\, dv && (6)\\
&= \frac{1}{2}\int \rho\phi\, dv, && (7)
\end{aligned}$$

where in 4 we used the equation $\mathbf{E} = \nabla\phi$, in 5 we used the identity given in the question and in 6 we used the divergence theorem and Gauss's law in differential form. As the charge density is local, at large distances from the charge $\phi \sim 1/r$ and $\nabla\phi = E \sim 1/r^2$. So we have $dA\phi E \sim 1/r$ as $dA \sim r^2$. So in the limit that $r \to 0$ the first term in 6 is zero.

Q2)

The forces on charges q_b and q_c are clearly zero because a conductor shields them from any external fields. The force on charge q_d however is not zero, in the case of r being large we have

$$\mathbf{F}_{q_d} = -\frac{(q_b + q_c)q_d}{r^2}\hat{i} \tag{8}$$

and A feels an equal and opposite force. Clearly these are the only two forces that depend on r large.

Q3)

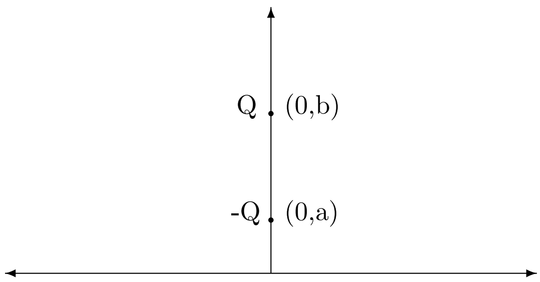

Fig. 1

We have two charges above the conduction plane so we can replace the conducting plane by two image charges below the conducting plane. Therefore the force on the negative charge is

$$F = -\frac{Q^2}{4b^2} + \frac{Q^2}{(b-a)^2} + \frac{Q^2}{(b+a)^2}. \tag{9}$$

So if $F = 0$ we have the quadratic equation

$$\begin{aligned} 7a^4 + 10b^2a^2 - b^4 &= 0 & (10) \\ a^2 &= \frac{-5 \pm 4\sqrt{2}}{7}b^2 & (11) \\ a &= \sqrt{\frac{-5 + 4\sqrt{2}}{7}}b & (12) \\ &= 3.06\text{cm}, & (13) \end{aligned}$$

where we set $b = 10$cm in final line.

Q4)

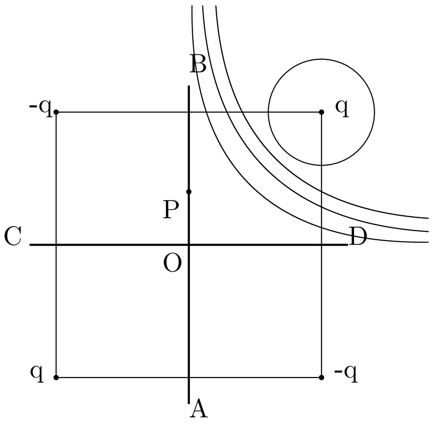

Fig. 2

Clearly by symmetry the lines AB and CD are equipotentials. To show that AD is an equipotential line let the origin be at O. Let the side of the square $= 2a$. Then the potential at the point $P \equiv (0, y)$ is then

$$\begin{aligned} \phi &= -\frac{q}{\sqrt{(a-y)^2+a^2}} + \frac{q}{\sqrt{(a-y)^2+a^2}} + \frac{q}{\sqrt{(a+y)^2+a^2}} - \frac{q}{\sqrt{(a+y)^2+a^2}} \\ &= 0. \end{aligned}$$

Similarly along CD the potential is $= 0$. The equipotential lines will have asymptotes at $x = 0$ and $y = 0$ and close to the charge it looks like a point charge potential. Now for a charge in between with two semi-infinite conducting planes forming a 90^0 angle we have image charges that look exactly like initial charge configuration. The method of images will work as long as the images of the charges converge to a point. This is exactly the diagonal point our configuration. However if the angle between the plates is obtuse then the images in each of planes do not converge. If one looks at the case of 120^0 the images form exactly on the reflection of one of the conducting surfaces. So each of the reflected surfaces keep getting mapped back to themselves.

Q5)

If we have Q on inner surface of the outer sphere and $-Q$ on the outer surface of the inner sphere and the corresponding radii are b and a respectively. Therefore the electric field inbetween them is $\mathbf{E} = -\frac{Q}{r^2}\hat{r}$. So the potential difference between the two surfaces $V = \frac{Q}{a} - \frac{Q}{b}$. Now using the relation $Q = CV$ we have

$$C = Q/V = \frac{ab}{b-a}. \tag{14}$$

In the limit of $b - a = s$ is small we have $C \to \frac{b^2}{s}$. Now from eqn. 3.16 we know $C = \frac{A}{4\pi s} = \frac{b^2}{s}$ as for the sphere $A = 4\pi b^2$.

Q6)

If we have two concentric infinite cylinders with the inner cylinder of outer radius a and outer cylinder of inner radius b. Then with charge $-\lambda$ and λ distributed on each of them the electric field between them using Gauss's law is $\mathbf{E} = -2\frac{\lambda}{r}\hat{r}$. So the potential between the two surfaces $= \lambda \log(\frac{b}{a})$ and if total charge $Q = \lambda h$is put on each surface then $C = \frac{h}{\log(\frac{b}{a})}$. We have $h = 30$cm, $a = 3$cm, $h = 4$cm. So the energy stored is $\frac{1}{2}CV^2$ and we know $V = 45\text{V} = 0.15\text{statVolt}$. So the energy stored $= 2.34$ ergs.

Q7)

We have a sphere of radius a and an inner sphere of radius b. From Q5 we know that the capacitance is $C = \frac{ab}{a-b}$ and the energy stored $U = \frac{1}{2}CV^2$ and the electric field on the inner sphere is held constant E_0. If Q is the total charge on the inner sphere then the potential difference across the spheres $V = Q(\frac{a-b}{ab})$ and the electric field at the surface of the inner sphere is $E_0 = \frac{Q}{b^2}$. So eleminating Q we have $V = \frac{(a-b)b}{a}E_0$. So the total energy is then $U = \frac{1}{2}\frac{(a-b)b^3}{a}E_0^2$. Therefore $\frac{dU}{db} = 0$ gives us that $3ab^2 - 4b^3 = 0$, so we fined $b = 3/4a$. The energy stored is then $U = 27/512a^3$.

Q8)

We have the initial values of the points label in the diagram are

$$a' = a = c' = c = e = 50 \tag{15}$$
$$b' = b = d = f' = f = g = 25 \tag{16}$$
$$\tag{17}$$

1st iteration:

$$a_1 = 1/4(100 + a' + b + c) = 56.25 \tag{18}$$
$$b_1 = 1/4(b' + d + a) = 25 \tag{19}$$
$$c_1 = 1/4(100 + a + d + e) = 56.25 \tag{20}$$
$$d_1 = 1/4(b + c + f) = 25 \tag{21}$$
$$e_1 = 1/4(c + f + c' + f') = 37.5 \tag{22}$$
$$f_1 = 1/4(d + e + g) = 25 \tag{23}$$
$$g_1 = 1/4(f + f') = 12.5, \tag{24}$$

and replacing again $a = a', c = c', f = f'$. 2nd iteration:

$$a_2 = 1/4(100 + a_1' + b_1 + c_1) = 59.38 \tag{25}$$
$$b_2 = 1/4(b_1' + d_1 + a_1) = 26.56 \tag{26}$$

$$c_2 = 1/4(100 + a_1 + d_1 + e_1) = 54.69 \tag{27}$$
$$d_2 = 1/4(b_1 + c_1 + f_1) = 26.56 \tag{28}$$
$$e_2 = 1/4(c_1 + f_1 + c_1' + f_1') = 40.63 \tag{29}$$
$$f_2 = 1/4(d_1 + e_1 + g_1) = 18.75 \tag{30}$$
$$g_2 = 1/4(f_1 + f_1') = 12.5, \tag{31}$$

and replacing again $a = a', c = c', f = f'$. 3rd iteration:

$$a_3 = 1/4(100 + a_2' + b_2 + c_2) = 60.16 \tag{32}$$
$$b_3 = 1/4(b_2' + d_2 + a_2) = 28.13 \tag{33}$$
$$c_3 = 1/4(100 + a_2 + d_2 + e_2) = 56.64 \tag{34}$$
$$d_3 = 1/4(b_2 + c_2 + f_2) = 25 \tag{35}$$
$$e_3 = 1/4(c_2 + f_2 + c_2' + f_2') = 36.72 \tag{36}$$
$$f_3 = 1/4(d_2 + e_2 + g_2) = 19.92 \tag{37}$$
$$g_3 = 1/4(f_2 + f'_{)} = 9.38, \tag{38}$$

and replacing again $a = a', c = c', f = f'$. 4th iteration:

$$a_4 = 1/4(100 + a_3' + b_3 + c_3) = 61.23 \tag{39}$$
$$b_4 = 1/4(b_3' + d_3 + a_3) = 28.32 \tag{40}$$
$$c_4 = 1/4(100 + a_3 + d_3 + e_3) = 55.47 \tag{41}$$
$$d_4 = 1/4(b_3 + c_3 + f_3) = 26.17 \tag{42}$$
$$e_4 = 1/4(c_3 + f_3 + c_3' + f_3') = 38.28 \tag{43}$$
$$f_4 = 1/4(d_3 + e_3 + g_3) = 17.77 \tag{44}$$
$$g_4 = 1/4(f_3 + f_3') = 9.96, \tag{45}$$

and replacing again $a = a', c = c', f = f'$. 4th iteration:

$$a_5 = 1/4(100 + a_4' + b_4 + c_4) = 61.25 \tag{46}$$
$$b_5 = 1/4(b_4' + d_4 + a_4) = 28.93 \tag{47}$$
$$c_5 = 1/4(100 + a_4 + d_4 + e_4) = 56.42 \tag{48}$$
$$d_5 = 1/4(b_4 + c_4 + f_4) = 25.39 \tag{49}$$
$$e_5 = 1/4(c_4 + f_4 + c_4' + f_4') = 36.62 \tag{50}$$
$$f_5 = 1/4(d_4 + e_4 + g_4) = 18.60 \tag{51}$$
$$g_5 = 1/4(f_4 + f_4') = 8.89. \tag{52}$$

Therefore find that the contours will be evenly spaced in the middle of the square, but they will be closely bunched at the inner corner and more spread out towards the outer corner.