

Physics 142: Homework 2

January 21, 2004

Q1)

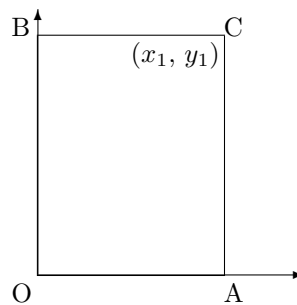


Fig. 1

We have the electric field is

$$\mathbf{E} = 6xy \hat{i} + (3x^2 - 3y^2) \hat{j} \quad (1)$$

and along OA and BC the line element is $\mathbf{ds} = dx \hat{i}$ while along OB and AC it is $\mathbf{ds} = dy \hat{j}$. So along the path OAC we have

$$\Phi = - \int_O^C \mathbf{E} \cdot \mathbf{ds} \quad (2)$$

$$= - \int_0^{x_1} (E_x)|_{y=0} dx - \int_0^{y_1} (E_y)|_{x=x_1} dy \quad (3)$$

$$= - \int_0^{x_1} 0 dx - \int_0^{y_1} (3x_1^2 - 3y^2) dy \quad (4)$$

$$= -3x_1^2 y_1 + y_1^3. \quad (5)$$

Now along OBC we have

$$\Phi = - \int_O^C \mathbf{E} \cdot \mathbf{ds} \quad (6)$$

$$= - \int_0^{y_1} (E_y)|_{x=0} dy - \int_0^{x_1} (E_x)|_{y=y_1} dx \quad (7)$$

$$= -\int_0^{y_1} (3x_1^2 - 3y^2)dy - \int_0^{x_1} 6xy_1 dx \quad (8)$$

$$= -3x_1^2 y_1 + y_1^3. \quad (9)$$

Since (x_1, y_1) is an arbitrary point we can find the electric field at (x_1, y_1) using

$$\mathbf{E} = -\nabla\Phi \quad (10)$$

$$= -\left(\frac{\partial\Phi}{\partial x_1}\hat{i} + \frac{\partial\Phi}{\partial y_1}\hat{j}\right) \quad (11)$$

$$= 6x_1 y_1 \hat{i} + 3(x_1^2 - y_1^2)\hat{j} \quad (12)$$

which is exactly the electric field we started with. So we verify that $\mathbf{E} = -\nabla\Phi$.

Q2)

We have

$$\phi = \begin{cases} x^2 + y^2 + z^2 & \text{when } x^2 + y^2 + z^2 < a^2 \\ -a^2 + \frac{2a^3}{(x^2 + y^2 + z^2)^{1/2}} & \text{when } x^2 + y^2 + z^2 > a^2 \end{cases} \quad (13)$$

So we have the electric field is

$$\mathbf{E} = -\nabla\phi \quad (14)$$

$$= -\begin{cases} 2x\hat{i} + 2y\hat{j} + 2z\hat{k} & \text{when } x^2 + y^2 + z^2 < a^2 \\ -\frac{2a^3}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{i} + y\hat{j} + z\hat{k}) & \text{when } x^2 + y^2 + z^2 > a^2 \end{cases} \quad (15)$$

and charge density is

$$\rho = \frac{1}{4\pi}\nabla \cdot \mathbf{E} \quad (16)$$

$$= -\frac{1}{4\pi} \begin{cases} 6 & \text{when } x^2 + y^2 + z^2 < a^2 \\ 0 & \text{when } x^2 + y^2 + z^2 > a^2 \end{cases} \quad (17)$$

We also know that $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so the electric field points radially inward everywhere and a charge density $-6/4\pi$ inside a sphere of radius a . So we have solid sphere with a total charge of $-2a^3$ distributed uniformly. However if we now look at the electric field outside it looks like that of a point charge with $q = 2a^3$. This implies that there is an additional charged shell with total charge $4a^3$. So the total charge distribution is a solid charged sphere with total charge $-2a^3$ surrounded by a spherical shell of total charge $4a^3$.

Q3)

If there is total charge Q on a sphere of radius r then the potential on the surface of the sphere is $V = \frac{Q}{r}$. So we have the total density charge on the

sphere $Q = \frac{V}{4\pi r}$. The electron density added to the sphere is then $n = Q/e$ where e is the charge of the electron. So we have

$$\text{Radius of a basketball } r = 20\text{cm} \quad (18)$$

$$V = 10^3 V = -\frac{10}{3}\text{statvolt} \quad (19)$$

$$e = -1.6 \times 10^{-19} C = -4.8 \times 10^{-10} \text{esu}. \quad (20)$$

So the electron density added $n = 2.8 \times 10^7$.

Q4)

a)

Gauss's law gives us the equation

$$\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi Q_{enc}. \quad (21)$$

In picture 2.17 we have a cylinder of radius a and uniform charge density ρ that is constant. For a Gaussian surface at a distance $r < a$ the amount of charge enclosed is $Q_{enc} = \pi r^2 h \rho$. Now due to the symmetry of the cylinder we also have $\oint \mathbf{E} \cdot d\mathbf{A} = \mathbf{E} \cdot d\mathbf{A} = EA$. The last equality holds because the electric field points radially outwards. Therefore substituting these results in 21 we have

$$E2\pi r h = 4\pi^2 r^2 h \rho \quad (22)$$

$$E = 2\pi r \rho. \quad (23)$$

When we are outside the cylinder all of the charged cylinder is enclosed. So $Q_{enc} = \pi a^2 h$. Therefore the electric field outside is

$$E2\pi r h = 4\pi^2 a^2 h \rho \quad (24)$$

$$E = 2\pi \rho \frac{a^2}{r} \quad (25)$$

So we have electric field is

$$\mathbf{E} = \begin{cases} 2\pi r \rho \hat{r} & \text{when } r < a \\ 2\pi \rho \frac{a^2}{r} \hat{r} & \text{when } r > a \end{cases} \quad (26)$$

b)

The electric potential at a position $r < a$ and $d\mathbf{s} = dr\hat{r}$ is

$$\Phi = - \int \mathbf{E} \cdot d\mathbf{s} \quad (27)$$

$$= -\pi \rho r^2, \quad (28)$$

where we have chosen $\Phi(0) = 0$. Similarly we have for $r > a$

$$\Phi = -2\pi\rho a^2 \log(r) + C. \quad (29)$$

where C is a constant. To find C we need that Φ be continuous at $r = a$. This implies $C = 2\pi\rho a^2 \log(a) - \pi\rho a^2$. Therefore the potential function

$$\Phi = \begin{cases} -\pi\rho r^2 & \text{when } r < a \\ -2\pi\rho a^2 \log(r/a) - \pi\rho a^2 & \text{when } r > a \end{cases} \quad (30)$$

Q5)

We have $f(x, y) = x^2 + y^2$ so we have $\nabla f = 2x\hat{i} + 2y\hat{j}$ and therefore $\nabla^2 f = 2 + 2 = 4$. So clearly $f(x, y)$ does not satisfy Laplace's equation.

For $g(x, y) = x^2 - y^2$ have that $\nabla g = 2x\hat{i} - 2y\hat{j}$ and therefore $\nabla^2 g = 2 - 2 = 0$. So $g(x, y)$ satisfies Laplace's equation. The sketch of the function corresponds to $g(x, y) = c$ where c is constant. Depending on c we can generate a family of curves and in Fig 2. we have sketched only 2 such curves.

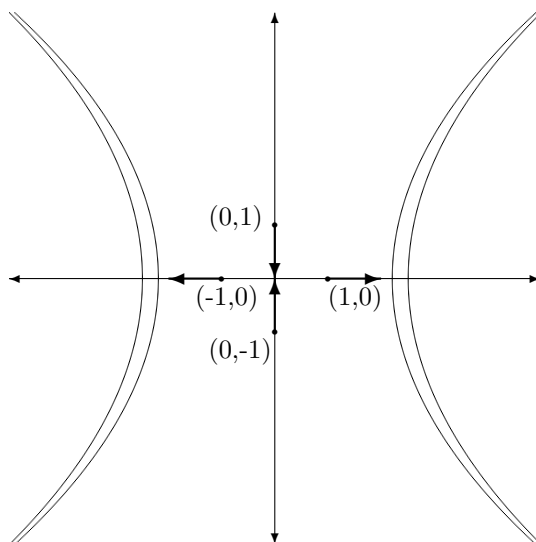


Fig. 2

From our formula of the gradient given above we have

$$\text{For } (x = 0, y = 1) : \nabla g = -2\hat{j} \quad (31)$$

$$\text{For } (x = 1, y = 0) : \nabla g = 2\hat{i} \quad (32)$$

$$\text{For } (x = 0, y = -1) : \nabla g = 2\hat{j} \quad (33)$$

$$\text{For } (x = -1, y = 0) : \nabla g = -2\hat{i}. \quad (34)$$

In Fig. 2 the vectors lengths have been scaled to half their actual values, so as to illustrate the direction of the gradient at each point.

Q6)

The potential energy stored in a charge spherical metal sphere with charge q and radius r is

$$U = \frac{q^2}{2r} \quad (35)$$

In this problem we have total charge to be distributed between two spheres with radii r_1 and r_2 respectively. Also these spheres are sufficiently separated so that the interaction between the two of them is negligible. Therefore the charge distribution is still spherically symmetric. Therefore we apply the above equation for the potential energy. Now let q be the charge on sphere 1 then the charge on sphere 2 is $Q - q$. Therefore the total energy of the system is then

$$U_{tot} = \frac{q^2}{2r_1} + \frac{(Q - q)^2}{2r_2} \quad (36)$$

To minimize the energy we need that $\frac{dU_{tot}}{dq} = 0$ which gives that

$$\frac{q}{r_1} - \frac{Q - q}{r_2} = 0 \quad (37)$$

$$\Rightarrow q = \frac{\frac{Q}{r_2}}{\frac{1}{r_1} + \frac{1}{r_2}} \quad (38)$$

$$= Q \frac{r_1}{r_1 + r_2}. \quad (39)$$

Therefore the potential of sphere 1 is then $= \frac{q}{r_1} = \frac{Q}{r_1 + r_2}$ and the potential of sphere 2 $= \frac{Q - q}{r_2} = \frac{Q}{r_1 + r_2}$. Therefore the potential difference is 0.

Q7)

a)

From eqn. 20 and 21 in Chapter 2. we know the potential of a uniformly charged disc along the symmetry axis is

$$\Phi = 2\pi\sigma \begin{cases} \sqrt{y^2 + a^2} + y & \text{when } y < 0 \\ \sqrt{y^2 + a^2} - y & \text{when } y > 0 \end{cases}, \quad (40)$$

where σ is the surface charge density and a is the radius of the disc. The potential of a disc of uniform charge σ and radius a with hole of radius b in it can then be found using the principle of superposition. So the total potential is sum of the potential due disc of radius a and charge density σ and the potential of disc with radius b and charge density $-\sigma$. Therefore the resultant potential is

$$\Phi = 2\pi\sigma \begin{cases} \sqrt{y^2 + a^2} - \sqrt{y^2 + b^2} & \text{when } y < 0 \\ \sqrt{y^2 + a^2} - \sqrt{y^2 + b^2} & \text{when } y > 0 \end{cases}. \quad (41)$$

So at the centre of the disc we have

$$\Phi = 2\pi\sigma(a - b) \quad (42)$$

So we find the potential at the centre of a disc of radius $a = 3$ cm and hole of radius $b = 1$ cm with charge $\sigma = -4$ esu/cm² is -50.3 statvolts

b)

The potential energy an electron of charge $e = -4.8 \times 10^{-10}$ esu and mass $m = 9.1 \times 10^{-28}$ g at the centre of the disc is $= e\Phi$. At infinity all the potential energy is converted into kinetic energy so the final velocity of the electron $v = \sqrt{\frac{2e\Phi}{m}} = 7.3 \times 10^9$ cm/s.

An alternative way of solving this question is to use the formula for potential

$$\phi = \int_a^b dq/r. \quad (43)$$