Spin Statistics Theorem

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Organization of this talk

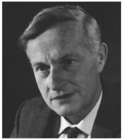
- Contents of ”Spin Statistics Theorem” and a little history
- A few heuristic methods to prove the theorem
- Elementary proof
- Understand the theorem in a topological way

the contents of spin statistics theorem - Wiki

- The wave functions of a system of identical integer-spin particles, spin 0, 1, 2, 3, has the same value when the positions of any two particles are exchanged. Particles with wave functions symmetric under exchange are called bosons.
- The wave functions of a system of identical half-integer-spin s = 1/2, 3/2, 5/2, are anti-symmetric under exchange, meaning that the wavefunction changes sign when the positions of any pair of particles are swapped. Particles whose wavefunction changes sign are called fermions.

A little history

- First formulated in 1939 by Markus Fierz
- Rederived in a more systematic way in 1940 by Wolfgang Pauli
- More conceptual argument was provided in 1950 by Julian Schwinger



A little history

Feynman Lectures on Physics: ...An explanation has been worked out by Pauli from complicated arguments of QFT and relativity...but we haven't found a way of reproducing his arguments on an elementary level...this probably means that we do not have a complete understanding of the fundamental principle involved...

Transition Amplitude

The transition amplitude to start with an initial state $|i>$ at time $-\infty$ and end with $|f>$ at time $+\infty$ is:

$$\mathcal{T} =< f|Texp[-i \int_{-\infty}^{+\infty} dtH_I(t)]|i>$$

where $\mathcal{T}$ is the time ordering symbol: a product of operators to its right is to be ordered not as written, but with operators at later times to the left of those at earlier times , and $H_I(t)$ is the perturbing hamiltonian in the *interaction picture*:

$$H_I(t) = exp(+iH_0t)H_1exp(-iH_0t)$$

*Key Point:*for the transition amplitude to be Lorentz Invariant, the time ordering must be frame independent!

The time ordering of 2 spacetime points x and x' is frame independent if their separation is timelike:$(x - x')^2 \lnsim 0$, but it could have different temporal ordering in different frames if their separation is spacelike:$(x - x')^2 \gnsim 0$. In order to avoid $\mathcal{T}$ being different in different frames, we must require:

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0, whenever (x - x')^2 \gnsim 0$$

where $\mathcal{H}_I(x)$ is the density of $H_I(x)$, and

$$\varphi(x) = \varphi^+(x) + \varphi^-(x)$$

$$[a(\mathbf{k}), a(\mathbf{k}')]_\mp = 0 \ ,$$
$$[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp = 0 \ ,$$
$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp = (2\pi)^3 2\omega \, \delta^3(\mathbf{k} - \mathbf{k}')$$

$$\varphi^+(\mathbf{x}, t) = \int \widetilde{dk} \; e^{ikx} \; a(\mathbf{k}) \ ,$$
$$\varphi^-(\mathbf{x}, t) = \int \widetilde{dk} \; e^{-ikx} \; a^\dagger(\mathbf{k})$$

Obviously, $[\varphi^+(x), \varphi^+(x')]_\mp = [\varphi^-(x), \varphi^-(x')]_\mp = 0$. However,

$$\begin{aligned}
[\varphi^+(x), \varphi^-(x')]_\mp &= \int \widetilde{dk}\, \widetilde{dk}'\; e^{i(kx-k'x')}[a(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp \\
&= \int \widetilde{dk}\; e^{ik(x-x')} \\
&= \frac{m}{4\pi^2 r} K_1(mr) \\
&\equiv C(r) \, .
\end{aligned}$$

To solve this problem, we redefine:

$$\begin{aligned}
\varphi_\lambda(x) &\equiv \varphi^+(x) + \lambda\varphi^-(x) \, , \\
\varphi_\lambda^\dagger(x) &\equiv \varphi^-(x) + \lambda^*\varphi^+(x) \, ,
\end{aligned}$$

where λ is an arbitrary complex number. We then have

$$\begin{aligned}
[\varphi_\lambda(x), \varphi_\lambda^\dagger(x')]_\mp &= [\varphi^+(x), \varphi^-(x')]_\mp + |\lambda|^2[\varphi^-(x), \varphi^+(x')]_\mp \\
&= (1 \mp |\lambda|^2)\, C(r)
\end{aligned}$$

and

$$\begin{aligned}
[\varphi_\lambda(x), \varphi_\lambda(x')]_\mp &= \lambda[\varphi^+(x), \varphi^-(x')]_\mp + \lambda[\varphi^-(x), \varphi^+(x')]_\mp \\
&= \lambda(1 \mp 1)\, C(r) \, .
\end{aligned}$$

The only way is to choose $|\lambda| = 1$, and choose commutators.
Similar thing happens to Spin-$\frac{1}{2}$ case!

By Gerhart Luders and Bruno Zumino (1958).
Pauli (1940) did the proof in the case of noninteracting fields. In the presence of interaction the theorem splits into 2 parts:

- Commutation relations between two operators of the same field. Minus BB, Plus FF.
- Commutation relations between different fields. Minus BB and BF, Plus FF.

sth. about commutations between different fields

- the above case is not the only possible one
- interactions can be constructed in a way that commutation relations between different fields is to a certain extent arbitrary(by author G.L.).
- these other possibilities can be obtained by means of one or more generalized *Klein transformations*.
- we only consider the same-field case here.

Five assumptions–Also spin-0 case

- 1. The theory is invariant w.r.t to the proper inhomogeneous Lorentz Group(4-D trans. but no reflection)
- 2. Two operators of the same field at spacelike-separated points either commute or anticommute(locality)
- 3. The vacuum is the state of lowest energy.
- 4. the metric of the Hilbert space is positive definite.
- 5. The vacuum is not identically annihilated by a field.

Assumption 1

The theory is invariant w.r.t to the proper inhomogeneous Lorentz Group

It follows that the expectation value $\langle\varphi(x)\varphi(y)\rangle_0$ in vacuum is an invariant of the difference 4-vector: $\xi_\mu = x_\mu - y_\mu$:

$$\langle\varphi(x)\varphi(y)\rangle_0 = f(\xi)$$

for spacelike ξ, $f(\xi)$ only depends on the invariant $\xi_\mu\xi^\mu$. From assumption 1 we get:

$$\langle[\varphi(x), \varphi(y)]\rangle_0 = 0, \xi spacelike$$

Assumption 2

Two operators of the same field at spacelike-separated points either commute or anticommute(locality)

we have two choices here:

$$[\varphi(x), \varphi(y)]_{\pm} = 0, \xi \, spacelike$$

However, if we choose $[\varphi(x), \varphi(y)]_{+} = 0, \xi \, spacelike$, then

$$\langle[\varphi(x), \varphi(y)]_{+}\rangle_0 = 0, \xi \, spacelike$$

Leading to:

$$\langle\varphi(x)\varphi(y)\rangle_0 = 0, \xi \, spacelike$$

Assumption 3

The vacuum is the state of lowest energy.

From this,

$$\langle\varphi(x)\varphi(y)\rangle_0 = 0, \xi \, \textit{spacelike}$$

holds not only for spacelike ξ but also for allξ by the method of *Analytic Continuation.*

Assumption 4

the metric of the Hilbert space is positive definite.

It allows one to get:

$$\varphi(x)\Omega = 0$$

where Ω is the physical vacuum.

Assumption 5

The vacuum is not identically annihilated by a field.
Thus the choose of anticommutator is untenable.
Similar proof applies to spin one-half case.
The theorem for non-Hermitian fields can also be proved in this method!

The proof only involves Lorentz Invariance, but no Relativistic QFT. Schwinger assumed that the kinematic part of the Lagrangian by itself determines the spin-statistics connection!

Sudarshan considers $(3+1)$-D spacetime and imposes 4 conditions on the kinematic part of the Lagrangian for a field:

- 1.derivable from a local L.I. field theory;
- 2.in the Hermitian basis $\phi = \phi^\dagger$;
- 3.at most linear in the first derivatives of the field;
- 4.bilinear in the field ϕ.

The kinematic terms in the Schwinger Lagrangian have the generic form:

$$\mathcal{L} = \frac{i}{2}(\phi_r \dot{\phi}_s - \dot{\phi}_r \phi_s) K^0_{rs} - \frac{i}{2} \sum_{j=1,2,3} (\phi_r \nabla_j \phi_s - \nabla_j \phi_r \phi_s) K^j_{rs} - \phi_r \phi_s M_{rs}$$

where r,s relate to the spin of the field, and K,M are corresponding matrices of the field. L can be written as:

$$\mathcal{L} = \sum_{rs} \phi_r \Lambda_{rs} \phi_s,$$

$$\Lambda = \frac{i}{2} K^0 \overleftrightarrow{\partial}_t - \frac{i}{2} K_j \overleftrightarrow{\partial}_j - M$$

Lagrangian must be invariant under the change of order of any two fields b/c the order of the fields is undefined a priori and must be irrelevant.

A property of $SO(3)$ group:

- Representations belonging to integral spin have a bilinear scalar product symmetric in the indices of the factors;
- Half-integral spin representations have antisymmetric scalar products.

As a consequence, if $\phi_r \Longleftrightarrow \phi_s$, the affected terms in L:

$$\phi_r \Lambda_{rs} \phi_s + \phi_s \Lambda_{sr} \phi_r \mapsto \pm \phi_s \Lambda_{rs} \phi_r \pm \phi_r \Lambda_{sr} \phi_s$$

+ for integral spin and - for half-integral spin.

Invariance of Lagrangian requires :

$$\Lambda_{sr} = \pm\Lambda_{rs}$$

Thus the matrix M must be symmetric for integral spin field, and antisymmetric for half-integral spin field.
BUT ???,

- Symmetric M corresponds to Bose-Einstein statistics.
- Antisymmetric M corresponds to Fermi-Dirac statistics.

Therefore,

- Integral spin $\Leftrightarrow$ Bose-Einstein statistics
- Half-integral spin $\Leftrightarrow$ Fermi-Dirac statistics

Understanding the theorem in a topological way

Where does the minus come from when:

- Rotating a spin one-half particle by 2π;
- Exchanging two spin-one-half particles?

Gould's argument

A 2π rotation is not just a trivial return of everything to what is was!
Place a cup full of coffee on your hand and rotate it!
Critique from Hilborn:

- Nowhere does the spin of the object enter the question
- Nor is it clear what the twist in the arm has to do with the change in sign of fermion's wave function!

Feynman's models-1 rotate a particle

Charge-Monopole composite

spin $\frac{1}{2}$ $\Leftrightarrow$ spin 0 electric charge e + spin 0 magnetic monopole of magnetic charge g

Its *EM* angular momentum

$$\vec{L} = \int \vec{r} \times (\vec{E} \times \vec{B}) d^3 r$$

is independent of the separation of e and g, directed along the line between them, and equal to *eg*, giving $eg = \frac{1}{2}$ when angular momentum assumes its minimum nonzero value.

Move e around g by 2π, the wave function acquires phase change

$$\phi = e \int \vec{A} \cdot d\vec{l} = e \int \vec{B} \cdot d\vec{s} = eg \times 2\pi = \pi$$

Feynman's models-2 exchange two particles

Exchange 2 eg composites, call them 1 at x and 2 at y.
view this exchange as

- 1 translated from x to y in the vector potential of 2;
- 2 translated from x to y in the vector potential of 1.

The total phase change is

$$\phi = \phi_1 + \phi_2 = e\int_x^y \vec{A}_2 \cdot d\vec{l}_1 + e\int_y^x \vec{A}_1 \cdot d\vec{l}_2 = e\int \vec{B} \cdot d\vec{s} = eg \times 2\pi = \pi$$

Critique of Feynman's models

- didn't view elementary particles as mathematical point and endowed it with a unphysical superstructure of magnetic field
- The exchange operation is the rigid rotation of each composite but with no apprent internal rotations.

Conclusion

- A brief review of 3 different ways to prove the spin-statistics theorem, one of which is regarded as elementary by some people;
- Try to understand the theorem from topological view.