PHYSICS 234 HOMEWORK 9 SOLUTIONS

Q1. We have the following differential equation

$$\frac{d^2v}{d\rho^2} - 2\frac{dv}{d\rho} + \left[\frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right] v = 0 \tag{1}$$

and substituting the power series expansion

$$v(\rho) = \sum_{k=0}^{\infty} C_k \rho^{k+l+1} \tag{2}$$

we find

$$\sum_{k=0}^{\infty}(k+l)(k+l+1)C_k\rho^{k+l-1} - 2\sum_{k=0}^{\infty}(k+l+1)C_k\rho^{k+l} \tag{3}$$

$$+e^2\lambda\sum_{k=0}^{\infty}C_k\rho^{k+l} - l(l+1)\sum_{k=0}^{\infty}C_k\rho^{k+l-1} = 0 \tag{4}$$

$$\Rightarrow \sum_{k=0}^{\infty}\left[(k+l)(k+l+1) - l(l+1)\right]C_k\rho^{k+l-1} - \tag{5}$$

$$\sum_{k=0}^{\infty}\left[2(k+l+1) - e^2\lambda\right]C_k\rho^{k+l} = 0 \tag{6}$$

$$\Rightarrow \sum_{k=1}^{\infty}\left[(k+l)(k+l+1) - l(l+1)\right]C_k\rho^{k+l-1} - \tag{7}$$

$$\sum_{k=1}^{\infty}\left[2(k+l) - e^2\lambda\right]C_{k-1}\rho^{k+l-1} = 0. \tag{8}$$

In the last step we change $k \to k+1$ in the second sum and the first term of the first sum is zero as when $k=0$ we have $(k+l)(k+l+1) = l(l+1)$. Therefore we find the recursion relation

$$\frac{C_{k+1}}{C_k} = \frac{2(k+l+1) - e^2\lambda}{(k+l+2)(k+l+1) - l(l+1)}. \tag{9}$$

Now in limit of large k the ratio grows as $2/k$, so as the book says the coefficient $C_k \sim \frac{2^k}{k!}$ and therefore $v \sim \rho^m e^{2\rho}$. So for $U \to 0$ as $x \to \infty$ we need that the series truncate. So we find that

$$e^2\lambda = 2(k+l+1) \tag{10}$$

$$e^2\sqrt{\frac{2m}{\hbar^2 W}} = 2(k+l+1) \tag{11}$$

$$\Rightarrow E = -W = -\frac{me^4}{2\hbar^2(k+l+1)} \tag{12}$$

Q2. We have to evaluate the sum

$$\sum_{l=0}^{n-1}(2l+1) = 2\sum_{l=0}^{n-1} l + n \tag{13}$$

$$= 2\frac{n(n-1)}{2} + n = n^2 \tag{14}$$

Q3. We have the difference in energy between the hydrogen and deutrium atoms is due to the masses of the nucleus being different. So as the mass parameter m that occurs in 12 is really the reduced mass we have the energy difference is

$$E_D - E_H = \frac{e^4}{2\hbar^2 n}(\mu_H - \mu_D) \tag{15}$$

where $\mu_D = \frac{2m_e m_p}{m_e+2m_p}$ and $\mu_H = \frac{m_e m_p}{m_e+m_p}$ are the reduced masses with m_e being the mass of the electron and m_p being the mass of a nucleon. Therefore the energy difference is

$$E_D - E_H = -\frac{e^4}{2\hbar^2 n}\left(\frac{2m_e m_p}{m_e + 2m_p} - \frac{m_e m_p}{m_e + 2m_p}\right) \tag{16}$$

$$\approx E_H \frac{m_e}{2m_p} \tag{17}$$

Q4. We have to find the radial wavefunctions for $n = 4$ and $n = 5$. The simplest way of doing this despite what I said in officer hours is to use the generating function. We know that

$$R_{nl} = e^{-\kappa r}(2\kappa r)^l L_{n-l-1}^{2l+1}(2\kappa r) \tag{18}$$

where $\kappa = \frac{1}{an}$. The Laguerre polynomials are generated by the following equations

$$L_{q-p}^{p}(z) = (-1)p\frac{d^p}{dz^p}L_q(z) \tag{19}$$

$$L_q(z)(z) = e^z\frac{d^q}{dz^q}(e^{-z}z^q). \tag{20}$$

So for $n = 4$ we have 4 functions to find $R_{40}, R_{41}, R_{42}, R_{43}$. So $n = 4, l = 0$ or $p = 1$ and $q = 4$ need

$$L_4(z) = e^z\frac{d^4}{dz^4}(e^{-z}z^4) \tag{21}$$

$$= z^4 - 16z^3 + 72z^2 - 96z + 24 \tag{22}$$

$$L_3^1(z) = -\frac{d}{dz}L_4(z) \tag{23}$$

$$= 4(z^3 - 12z^2 + 36z - 24). \tag{24}$$

Similarly $n = 4, l = 1$ or $p = 3$ and $q = 5$

$$L_5(z) = e^z\frac{d^5}{dz^5}(e^{-z}z^5) \tag{25}$$

$$= -z^5 + 25z^4 - 200z^3 + 600z^2 - 600z + 120 \tag{26}$$

$$L_2^3(z) = -\frac{d^3}{dz^3}L_5(z) \tag{27}$$

$$= -60(z^2 - 10z - 20). \tag{28}$$

Similarly $n = 4, l = 2$ or $p = 5$ and $q = 6$

$$L_6(z) = e^z \frac{d^6}{dz^6}(e^{-z} z^6) \tag{29}$$
$$= z^6 - 36z^5 + 450z^4 - 2400z^3 + 5400z^2 - 4320z + 720 \tag{30}$$
$$L_5^1(z) = -\frac{d^5}{dz^5} L_6(z) \tag{31}$$
$$= 720(z - 6). \tag{32}$$

Similarly $n = 4, l = 3$ or $p = 7$ and $q = 7$

$$L_7(z) = e^z \frac{d^7}{dz^7}(e^{-z} z^7) \tag{33}$$
$$= -z^7 + 49z^6 - 882z^5 + 7350z^4 - 29400z^3 + 52920z^2 \tag{34}$$
$$-35280z + 5040 \tag{35}$$
$$L_7^0(z) = -\frac{d^7}{dz^7} L_7(z) \tag{36}$$
$$= -5040. \tag{37}$$

Similarly we can generate all $n = 5$ polynomials. Clearly all these wavefunctions with $l \neq 1$ have a node at 0 due to the factor or r^l

Q5. The bohr radius for any hydrogen like atom with one electron and nuclear charge Ze is just $a_Z = \frac{\hbar^2}{m_e Z e^2} = \frac{a_B}{Z}$. In this case we have the $Z_{Tritium} = 1$ and $Z_{Helium} = 2$. The groundstate wavefunction of any hydrogen-like atom is

$$\psi(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_Z^3}} e^{-r/a_Z} \tag{38}$$

So the projection of the groundstate of Tritium on the groundstate of Helium is

$$\langle \psi_{100}^{He+} | \psi_{100}^{T} \rangle = \int_0^\infty 4\pi r^2 dr \psi_{100}^{He+} \psi_{100}^{T} \tag{39}$$
$$= \frac{4}{\sqrt{a_1^3 a_2^3}} \int_0^\infty r^2 e^{-(1/a_1 + 1/a_2) r} dr \tag{40}$$
$$= \frac{4}{\sqrt{a_1^3 a_2^3}} \frac{2}{(1/a_1 + 1/a_2)^3} \tag{41}$$
$$= \frac{8 a_1^{3/2} a_2^{3/2}}{(a_1 + a_2)^3} = \frac{8\sqrt{8}}{27} \tag{42}$$

So the probability of being the in the ground state of Helium is $\frac{512}{729}$.

Q6. In this section we just have to repeat what we did in Q5 using the appropriate wavefunctions. Clearly the integrals for 2p state are zero because the Helium is spherically symmetric and so is proportional to Y_{00} while the initial 2p state proportional to $Y_{10}, Y_{1,\pm 1}$.

Q7. We use the virial theorem to find that

$$2\langle T \rangle = -\langle U \rangle \tag{43}$$

So using the fact that the total energy is $E = T + U$ we find that $\langle U \rangle = 2\langle E \rangle$. Now in the n^{th} energy eigenstate we have the expectation value of E is just $\frac{-Z^2 me^4}{2\hbar^2 n^2}$ and therefore $\langle U \rangle = \frac{-Z^2 me^4}{\hbar^2 n^2}$