PHYSICS 234 HOMEWORK 5 SOLUTIONS

Q1. For $x > 0$ we have exactly the same differential equation as the normal harmonic oscillator. The only difference in this case is with the boundary conditions. In this case we have the condition that the wavefunction should disappear at $x = 0$. This force half of our eigenfunctions be invalid. The only wavefunctions that survive are the odd ones $|2n+1\rangle$ and so the allowed energies are $2n+1+1/2 = (4n+3)/2$.

Q2. The ground state wavefunction is

$$\psi(x) = A \exp\left[-\frac{m\omega x^2}{2\hbar}\right]. \tag{1}$$

So normalizing the wavefunction we have

$$\begin{aligned}
1 &= |A|^2 \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx && (2)\\
&= |A|^2 \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega x^2}{\hbar}\right] dx && (3)\\
&= |A|^2 \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} \exp[-y^2]dx && (4)\\
&= |A|^2 \sqrt{\frac{\hbar\pi}{m\omega}} && (5)\\
\Rightarrow A &= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} && (6)
\end{aligned}$$

The harmonic oscillator potential is

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{7}$$

The classically forbidden region corresponds to when the particles energy is less than the potential energy. The region this correponds to for the ground state is when

$$\begin{aligned}
\frac{1}{2}m\omega^2 x_c^2 &= \frac{1}{2}\hbar\omega && (8)\\
\Rightarrow x_c &= \sqrt{\frac{\hbar}{m\omega}} && (9)
\end{aligned}$$

So the probability of being in the non-classical region is

$$P_{quantum} = \int_{-\infty}^{-x_c} |\psi(x)|^2 dx + \int_{x_c}^{\infty} |\psi(x)|^2 dx \tag{10}$$

$$= 2\int_{x_c}^{\infty} |\psi(x)|^2 dx \tag{11}$$

$$= 2\int_{\sqrt{\frac{\hbar}{m\omega}}}^{\infty} \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \exp\left[-\frac{m\omega x^2}{\hbar}\right] dx \tag{12}$$

$$= 2\frac{1}{\sqrt{\pi}}\int_{1}^{\infty} e^{-y^2} dy \tag{13}$$

$$= 1 - Erf(1) = 0.1573 \tag{14}$$

Q3. We have the harmonic oscillator state

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle e^{(-i\omega t)/2} + |1\rangle e^{(-3i\omega t)/2}\right). \tag{15}$$

So we find that

$$\langle E\rangle = \langle H\rangle = \langle\psi(t)|\hbar\omega(a^\dagger a + \frac{1}{2})|\psi(t)\rangle \tag{16}$$

$$= \frac{1}{2}\hbar\omega\left(\frac{1}{2} + \frac{3}{2}\right) = \hbar\omega \tag{17}$$

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\psi(t)|(a^\dagger + a)|\psi(t)\rangle \tag{18}$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(\langle 0|a|1\rangle e^{-i\omega t} + \langle 1|a^\dagger|0\rangle e^{i\omega t}) \tag{19}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t \tag{20}$$

$$\langle p\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle\psi(t)|(a^\dagger - a)|\psi(t)\rangle \tag{21}$$

$$= \frac{1}{2}i\sqrt{\frac{m\omega\hbar}{2}}(\langle 0|a|1\rangle e^{-i\omega t} - \langle 1|a^\dagger|0\rangle e^{i\omega t}) \tag{22}$$

$$= \sqrt{\frac{m\omega\hbar}{2}}\sin\omega t. \tag{23}$$

So we see that in the phase plane $p - x$ the expectation values move in an ellipse, while the energy is just the average of the ground and first excited energies.

4. The probability density for this state is then just

$$P(x) = |\psi(x,t)|^2 \tag{24}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\exp\left[-\frac{m\omega x^2}{\hbar}\right] + \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\frac{m\omega x^2}{\hbar}\exp\left[-\frac{m\omega x^2}{\hbar}\right] +$$

$$\left(\frac{2m\omega}{\hbar\pi}\right)^{1/2}\sqrt{\frac{m\omega}{\hbar}}x\exp\left[-\frac{m\omega x^2}{\hbar}\right]\cos\omega t \tag{25}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\exp\left[-\frac{m\omega x^2}{\hbar}\right]\left(1 + \frac{m\omega x^2}{\hbar} + \sqrt{\frac{2m\omega}{\hbar}}x\cos\omega t\right) \tag{26}$$

5. We have the Hamiltonian

$$H = \epsilon_1 a^\dagger a + \epsilon_2 (a + a^\dagger), \tag{27}$$

as $H = H^\dagger$ both ϵ_1 and ϵ_2 are real. Now let us consider the standard hamiltonian and position operators

$$H_0 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{28}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{29}$$

in which case we find that

$$\begin{aligned}
H &= \frac{\epsilon_1}{\hbar\omega} H_0 - \frac{\epsilon_1}{2} + \epsilon_2 \sqrt{\frac{2m\omega}{\hbar}} x && (30)\\
&= \frac{\epsilon_1}{\hbar\omega}\left(\frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2\right) - \frac{\epsilon_1}{2} + \epsilon_2\sqrt{\frac{2m\omega}{\hbar}} x && (31)\\
&= \frac{\epsilon_1}{\hbar\omega}\left(\frac{P^2}{2m} + \frac{1}{2} m\omega^2 \left(x^2 + 2\frac{\epsilon_2}{\epsilon_1}\sqrt{\frac{2\hbar}{m\omega}} x\right)\right) - \frac{\epsilon_1}{2} && (32)\\
&= \frac{\epsilon_1}{\hbar\omega}\left(\frac{P^2}{2m} + \frac{1}{2} m\omega^2 \left(x + \frac{\epsilon_2}{\epsilon_1}\sqrt{\frac{2\hbar}{m\omega}}\right)^2\right) - \frac{\epsilon_2^2}{\epsilon_1} - \frac{\epsilon_1}{2}. && (33)
\end{aligned}$$

So changing variable from x to $y = x + \frac{\epsilon_2}{\epsilon_1}\sqrt{\frac{2\hbar}{m\omega}}$ we have

$$\begin{aligned}
H &= \frac{\epsilon_1}{\hbar\omega}\left(\frac{P_y^2}{2m} + \frac{1}{2} m\omega^2 y^2\right) - \frac{\epsilon_1}{2} - \frac{\epsilon_2^2}{\epsilon_1} && (34)\\
&= \epsilon_1 b^\dagger b - \frac{\epsilon_2^2}{\epsilon_1} && (35)
\end{aligned}$$

where $b^\dagger$ is the creation operator of the new harmonic oscillator. Infact you can show that $b = a + \frac{\epsilon_2}{\epsilon_1}$. So the energy in this case will be $E_n = \epsilon_1 n - \frac{\epsilon_2^2}{\epsilon_1}$ and the eigenstates would be same as the usual harmonic oscillator except they would all be centered at the point $x = \frac{\epsilon_2}{\epsilon_1}\sqrt{\frac{2\hbar}{m\omega}}$