

PHYSICS 234 HOMEWORK 4 SOLUTIONS

Q1. From eqn. 5.1.3 we have the eigenvalue equation

$$(1) \quad H|E\rangle = \frac{P^2}{2m}|E\rangle = E|E\rangle$$

(2)

So now using the completeness relation we have

$$(3) \quad \int \frac{P^2}{2m}|x\rangle\langle x|E\rangle dx = E \int |x\rangle\langle x|E\rangle dx$$

$$(4) \quad \Rightarrow \int \langle x'| \frac{P^2}{2m} |x\rangle\langle x|E\rangle dx = E \int \langle x'|x\rangle\langle x|E\rangle dx$$

$$(5) \quad \Rightarrow -\frac{\hbar^2}{2m} \int \delta(x-x') \frac{d^2}{dx^2} \psi_E(x) dx = E \int \delta(x-x') \psi_E(x) dx$$

$$(6) \quad \Rightarrow \frac{d^2}{dx^2} \psi_E(x) = -\frac{2mE}{\hbar^2} \psi_E(x)$$

which is exactly the equation for a harmonic oscillator. Therefore the wave function for the free particle is

$$(7) \quad \psi_E(x) = \alpha e^{ix\sqrt{\frac{2mE}{\hbar^2}}} + \beta e^{-ix\sqrt{\frac{2mE}{\hbar^2}}}.$$

If $E < 0$ then we will have real decaying exponentials and so on the interval $\{-\infty, \infty\}$ these are not square-integrable and so cannot be part of the hilbert space.

Q2. We have the propagator as

$$(8) \quad U(t) = \exp \left[\frac{i}{\hbar} \left(\frac{\hbar^2 t}{2m} \frac{d^2}{dx^2} \right) \right]$$

$$(9) \quad = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar t}{2m} \right)^n \frac{d^{2n}}{dx^{2n}}$$

and the initial state of the free particle is

$$(10) \quad \psi(x, 0) = \frac{1}{\pi^{1/4}} \exp \left[-\frac{x^2}{2} \right]$$

$$(11) \quad = \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! 2^n}.$$

So using our formalism we have

$$(12) \quad \psi(x, t) = U(t)\psi(x, 0)$$

$$(13) \quad = \frac{1}{\pi^{1/4}} \sum_{n,k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar t}{2m} \right)^k \frac{d^{2k}}{dx^{2k}} \frac{(-1)^n x^{2n}}{n! 2^n}$$

$$(14) \quad = \frac{1}{\pi^{1/4}} \sum_{k=0, n=k}^{\infty} \frac{1}{k!} \left(\frac{i\hbar t}{2m} \right)^k \frac{2n!}{2(n-k)!} \frac{(-1)^n x^{2n-2k}}{n! 2^n}$$

where in the last line the first $2k-1$ terms are zero after being hit by $2k$ derivative operators. Now we relabel n by $j = n - k$ so we find that

$$(15) \quad \psi(x, t) = \frac{1}{\pi^{1/4}} \sum_{k=0, j=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar t}{2m} \right)^k \frac{2(j+k)!}{2j!} \frac{(-1)^{j+k} x^{2j}}{(j+k)! 2^{j+k}}$$

$$(16) \quad = \frac{1}{\pi^{1/4}} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!} \left(\sum_{k=0}^{\infty} \frac{1}{2^{j+k} k!} \left(\frac{-i\hbar t}{2m} \right)^k \frac{2(j+k)!}{(j+k)! 2j!} \right)$$

Now let us look at the coefficient in the sum over k

$$(17) \quad \frac{1}{2^{j+k} k!} \frac{2(j+k)!}{(j+k)! 2j!} j!$$

$$(18) \quad = \frac{1}{2^{j+k} k!} \frac{2^{j+k} (2j+2k-1) \cdots 1}{(2j-1) \cdots 1}$$

$$(19) \quad = \frac{1}{k!} (2j+2k-1) \cdots (2j+1)$$

$$(20) \quad = \frac{1}{k!} 2^k \left(j + \frac{2k-1}{2} \right) \cdots (j+1/2)$$

so then we find

$$(21) \quad \psi(x, t) = \frac{1}{\pi^{1/4}} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!} \left(\sum_{k=0}^{\infty} \left(\frac{-i\hbar t}{2m} \right)^k \frac{2^k}{k!} \left(j + \frac{2k-1}{2} \right) \cdots (j+1/2) \right)$$

$$(22) \quad = \frac{1}{\pi^{1/4}} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!} \left(1 + \frac{i\hbar t}{m} \right)^{-j-1/2}$$

$$(23) \quad = \left[\pi^{1/2} \left(1 + \frac{i\hbar t}{m} \right) \right]^{-1/2} \sum_{j=0}^{\infty} \left(\frac{-x^2}{\left(1 + \frac{i\hbar t}{m} \right)} \right)^j \frac{1}{j!}$$

$$(24) \quad = \left[\pi^{1/2} \left(1 + \frac{i\hbar t}{m} \right) \right]^{-1/2} \exp \left[\frac{-x^2}{\left(1 + \frac{i\hbar t}{m} \right)} \right]$$

which is exactly the result we find in the book when $p_0 = 0$, $\Delta = 1$ and $t' = 0$.

Q3. We have the initial state of the particle is the ground state of symmetric well where

$$(25) \quad V = \begin{cases} 0 & -L/2 \leq x \leq L/2 \\ \infty & \text{otherwise} \end{cases} .$$

So the ground state wavefunction is

$$(26) \quad \psi_i(x) = \sqrt{\frac{2}{L}} \cos \left(\frac{\pi x}{L} \right)$$

Now if the box suddenly expands to twice its earlier width the ground state wavefunction is then

$$(26) \quad \psi_0(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right)$$

and the initial state is now

$$(27) \quad \psi_i(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) & -L/2 \leq x \leq L/2 \\ 0 & \text{otherwise} \end{cases}.$$

So the ground state wavefunction is

So the component of the initial wavefunction along the ground state is

$$(28) \quad \langle \psi_0 | \psi_i \rangle = \frac{\sqrt{2}}{L} \int_{-L/2}^{L/2} \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{\pi x}{L}\right) dx$$

$$(29) \quad = \frac{\sqrt{2}}{L} \int_0^{L/2} \left(\cos\left(\frac{3\pi x}{2L}\right) + \cos\left(\frac{\pi x}{2L}\right) \right) dx$$

$$(30) \quad = \frac{\sqrt{2}}{L} \left(\frac{2L}{3\pi} \sin\left(\frac{3\pi}{4}\right) + \frac{2L}{\pi} \sin\left(\frac{\pi}{4}\right) \right)$$

$$(31) \quad = \frac{8}{3\pi}.$$

So the probability of being in the ground state = $|\langle \psi_0 | \psi_i \rangle|^2 = \left(\frac{8}{3\pi}\right)^2$.

Q4. We have the potential is

$$(32) \quad V(x) = -a\delta(x)$$

so for all points except 0 we have that

$$(33) \quad -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

Now as we are looking for bound state the energy must be negative and therefore the general solution to eqn. 33 is

$$(34) \quad \psi(x) = A \exp\left(\sqrt{\frac{-2mE}{\hbar^2}} x\right) + B \exp\left(-\sqrt{\frac{-2mE}{\hbar^2}} x\right).$$

To have this state be part of the hilbert space it has to be square integrable and since the positive exponential divergence for large positive x and the negative exponential for negative x we have

$$(35) \quad \psi(x) = \begin{cases} A \exp\left(\sqrt{\frac{-2mE}{\hbar^2}} x\right) & x < 0 \\ A \exp\left(-\sqrt{\frac{-2mE}{\hbar^2}} x\right) & x > 0 \end{cases}$$

$$(36) \quad = \begin{cases} A \exp(\kappa x) & x < 0 \\ A \exp(-\kappa x) & x > 0 \end{cases},$$

where $\kappa = \sqrt{\frac{-2mE}{\hbar^2}}$. We have implicit force the wavefunction to be continuous everywhere and especially 0 by making the constants be the same, namely A . Now

let us put this equation into the schrodinger equation and integrate over a region $\{-\epsilon, \epsilon\}$ and then take the limit as $\epsilon \rightarrow 0$ so that

$$(37) \quad \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) dx = E \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \psi dx$$

$$(38) \quad \Rightarrow -\lim_{\epsilon \rightarrow 0} \frac{\hbar^2}{2m} \left(\frac{d\psi(\epsilon)}{dx} - \frac{d\psi(-\epsilon)}{dx} \right) - aV_0\psi(0) = 0$$

$$(39) \quad \Rightarrow \frac{\kappa\hbar^2}{m} = aV_0$$

$$(40) \quad \Rightarrow \kappa = \frac{maV_0}{\hbar^2}$$

$$(41) \quad \Rightarrow E = -\frac{ma^2V_0^2}{2\hbar^2}$$

0.1. **Q5.** We have from eqn 5.3.8 that

$$(42) \quad \mathbf{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

or in one dimension we have

$$(43) \quad j = \frac{\hbar}{2mi} \left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right).$$

So for our wavefunction

$$(44) \quad \psi = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$$

we have

$$(45) \quad j = \frac{p}{2m} [(Ae^{ipx/\hbar} - Be^{-ipx/\hbar})(A^*e^{-ipx/\hbar} + B^*e^{ipx/\hbar}) -$$

$$(46) \quad (Ae^{ipx/\hbar} + Be^{-ipx/\hbar})(-A^*e^{-ipx/\hbar} + B^*e^{ipx/\hbar})]$$

$$(47) \quad = \frac{p}{m} (|A|^2 - |B|^2)$$

Q6.

a. We have exactly the same equation as in Q4, except in this case as particles are free $E > 0$. So we find the wavefunction is

$$(48) \quad \psi(x) = \begin{cases} A \exp^{ikx} + B \exp^{-ikx} & x < 0 \\ C \exp^{ikx} & x > 0 \end{cases},$$

where the left going amplitude is zero as we assume the wave is coming from the left. Continuity at $x = 0$ gives us $A + B = C$. Now clearly the the derivative of the wavefunction is not continuous as in Q4. So following the same procedure as we did in Q4 we have

$$(49) \quad -\lim_{\epsilon \rightarrow 0} \frac{\hbar^2}{2m} \left(\frac{d\psi(\epsilon)}{dx} - \frac{d\psi(-\epsilon)}{dx} \right) + aV_0\psi(0) = 0$$

the only difference being the sign of V_0 . So we find that

$$(50) \quad \frac{\hbar^2}{2m} ik(C - A + B) = aV_0(A + B)$$

$$(51) \quad \Rightarrow 2B = \frac{2maV_0}{ik\hbar^2}(A + B)$$

$$(52) \quad \Rightarrow \frac{B}{A} = \frac{\frac{-2imaV_0}{k\hbar^2}}{1 + \frac{2imaV_0}{k\hbar^2}}$$

$$(53) \quad \Rightarrow R = \frac{|B|^2}{|A|^2} = \frac{\left(\frac{2maV_0}{k\hbar^2}\right)^2}{1 + \left(\frac{2maV_0}{k\hbar^2}\right)^2}$$

$$(54)$$

and then the transmission rate using the fact that $R + T = 1$ is

$$(55) \quad T = \frac{1}{1 + \left(\frac{2maV_0}{k\hbar^2}\right)^2}$$

b. The potential is

$$(56) \quad V = \begin{cases} 0 & |x| > a \\ V_0 & |x| < a \end{cases}$$

So our wavefunction is then

$$(57) \quad \psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Ce^{\kappa x} + De^{-\kappa x} & |x| < a \\ Ee^{ikx} & x > a \end{cases}$$

Continuity and differentiability at $x = \pm a$ gives use the four equations

$$(58) \quad Ae^{-ika} + Be^{-ika} = Ce^{-\kappa a} + De^{\kappa a}$$

$$(59) \quad \frac{ik}{\kappa}(Ae^{-ika} - Be^{-ika}) = Ce^{-\kappa a} - De^{\kappa a}$$

$$(60) \quad Ce^{\kappa a} + De^{-\kappa a} = Ee^{ika}$$

$$(61) \quad Ce^{\kappa a} - De^{-\kappa a} = \frac{ik}{\kappa}Ee^{ika}.$$

Adding and subtracting eqns 58 with 59 and eqns 60 with 61 gives that

$$(62) \quad \left(1 + \frac{ik}{\kappa}\right)Ae^{-ika} + \left(1 - \frac{ik}{\kappa}\right)Be^{ika} = 2Ce^{-\kappa a}$$

$$(63) \quad \left(1 - \frac{ik}{\kappa}\right)Ae^{-ika} + \left(1 + \frac{ik}{\kappa}\right)Be^{ika} = 2De^{\kappa a}$$

$$(64) \quad \left(1 + \frac{ik}{\kappa}\right)Ee^{ika} = 2Ce^{\kappa a}$$

$$(65) \quad \left(1 - \frac{ik}{\kappa}\right)Ee^{ika} = 2De^{-\kappa a}.$$

Now eliminating C and D in the equations we find

$$(66) \quad \left(1 + \frac{ik}{\kappa}\right)Ae^{-ika} + \left(1 - \frac{ik}{\kappa}\right)Be^{ika} = \left(1 + \frac{ik}{\kappa}\right)Ee^{ika} e^{-2\kappa a}$$

$$(67) \quad \left(1 - \frac{ik}{\kappa}\right)Ae^{-ika} + \left(1 + \frac{ik}{\kappa}\right)Be^{ika} = \left(1 - \frac{ik}{\kappa}\right)Ee^{ika} e^{2\kappa a}.$$

Multiplying eqn 66 by $(1 - \frac{ik}{\kappa})$ and eqn 67 by $(1 + \frac{ik}{\kappa})$ and subtracting the two so as to eliminate B we find that

$$(68) \quad \left(1 + \frac{ik}{\kappa}\right)^2 + \left(1 - \frac{ik}{\kappa}\right)^2 A = \left(1 + \frac{ik}{\kappa}\right)^2 e^{-2\kappa a} + \left(1 - \frac{ik}{\kappa}\right)^2 e^{2\kappa a} E.$$

Therefore we find that

$$(69) \quad T = \frac{|E|^2}{|A|^2} = \left| \frac{\left(1 + \frac{ik}{\kappa}\right)^2 + \left(1 - \frac{ik}{\kappa}\right)^2}{\left(1 + \frac{ik}{\kappa}\right)^2 e^{-2\kappa a} + \left(1 - \frac{ik}{\kappa}\right)^2 e^{2\kappa a}} \right|^2$$

$$(70) \quad = \frac{\left(1 - \frac{k^2}{\kappa^2}\right)^2}{\left(1 - \frac{k^2}{\kappa^2}\right)^2 \cosh^2(2\kappa a) + \frac{4k^2}{\kappa^2} \sinh^2(2\kappa a)}$$

$$(71) \quad = \frac{\left(1 - \frac{k^2}{\kappa^2}\right)^2}{\left(1 - \frac{k^2}{\kappa^2}\right)^2 + \left(1 + \frac{k^2}{\kappa^2}\right)^2 \sinh^2(2\kappa a)}$$

The reflection coefficient is then just as $R = 1 - T$.

Q7. The quantum state in Appendix A is

$$(72) \quad |\psi\rangle = \frac{1}{2}|B_L B_R\rangle - \sqrt{\frac{3}{8}}(|B_L G_R\rangle + |B_L B_R\rangle)$$

So the probability you have both bad cakes is $1/4$.