PHYSICS 234 HOMEWORK 3 SOLUTIONS

Q1. First of all we assume Ω is an n dimensional operator that can be diagonalized, or in other words there exist n eigenvectors $|e_i\rangle$ whose j^{th} component is $|e_i\rangle_j$ where $1 \leq i, j \leq n$. Now let us consider the matrix U formed from $|e_i\rangle$ as the columns, so that

$$U_{ji} = |e_i\rangle_j. \tag{1}$$

The eigenvalue equation gives us

$$\Omega|e_i\rangle = e_i|e_i\rangle \tag{2}$$

$$\Rightarrow \sum_k \Omega_{jk}|e_i\rangle_k = e_i|e_i\rangle_j \tag{3}$$

$$\Rightarrow \sum_k \Omega_{jk}U_{ki} = e_i U_{ji}. \tag{4}$$

So now if we consider the equation

$$(U^\dagger \Omega U)_{ij} = \sum_{kl} U^{-1}_{ik}\Omega_{kl}U_{lj} \tag{5}$$

$$= \sum_k U^{-1}_{ik} e_j U_{kj} \tag{6}$$

$$= e_j \delta_{ij} \tag{7}$$

where in eqn. 6 we have used eqn. 4. So the matrix Ω is diagonalized by U whoes columns are made from Ω's eigenvectors using the usual similarity transformation $U^{-1}\Omega U$.

Q2. Using the same notation as we did in Q1. we have

$$U_{ij} = |v\rangle_i \langle u|_j = |v\rangle_i |u\rangle^*_j \tag{8}$$

and therefore the trace of U is

$$Tr(U) = \sum_i U_{ii} = \sum_i |v\rangle_i |u\rangle^*_i = \langle u|v\rangle \tag{9}$$

Q3. As H is a positive definitive hermitian matrix its eigenvalues $h_i \geq 0$ for all i. Now the trace of any matrix is invariant under a similarity transformations and therefore from the diagonal form $Tr(H) = \sum_i h_i$ which has to be greater than 0.

Q4.

a). To impose a vector space structure on $V = \{$Set of all complex $n \times n$ matrices $\}$ we need to define the operations of addition and scalar multiplication. For this space the obvious operations are the usual matrix addition and scalar multiplication in which each component is multiplied by the scalar. With these definitions the zero vector is the zero matrix and the additive inverse of say a vector U_{ij} is $-U_{ij}$. There are a total of n rows and n columns so the total number of independent elements is n^2.

b). We have two matrices A and B and if the inner product is defined as

$$\langle A|B\rangle = Tr(A^\dagger B) = \sum_{ij} A^*_{ij} B_{ij}. \tag{10}$$

So the first condition is satisfied because

$$\begin{aligned} \langle A|B\rangle &= Tr(A^\dagger B) && (11)\\ &= \sum_{ij} A^*_{ij} B_{ij} && (12)\\ &= (\sum_{ij} B^*_{ij} A_{ij})^* = \langle B|A\rangle^*, && (13) \end{aligned}$$

the second condition is satisfied as

$$\langle A|A\rangle = Tr(A^\dagger A) = \sum_{ij} |A_{ij}|^2 \geq 0 \tag{14}$$

and the third condition is true just because of the trace being distributive.

c). For $n = 2$ a general state is

$$\begin{aligned} (15)\ A &= \begin{pmatrix} a+ib & c+id \\ e+if & g+ih \end{pmatrix} \\ &= \frac{(a+g)+i(b+h)}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(a-g)+i(b-h)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \\ (16)\qquad & \frac{(c+e)+i(f+d)}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{(f-d)+i(c-e)}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ (17)\qquad &= \frac{(a+g)+i(b+h)}{2}1 + \frac{(a-g)+i(b-h)}{2}\sigma_x + \\ (18)\qquad & \frac{(c+e)+i(f+d)}{2}\sigma_y + \frac{(f-d)+i(c-e)}{2}\sigma_z. \end{aligned}$$

Therefore every 2×2 complex matrix can be expressed as a linear superposition of the $\{1, \sigma_x, \sigma_y, \sigma_z\}$. So they form a basis as they clearly linearly independent.

d). We have

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z \tag{19}$$

$$\sigma_y\sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x \tag{20}$$

$$\sigma_z\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y \tag{21}$$

so we find that $\sigma_\alpha\sigma_\beta = i\sigma_\gamma$. As if we switch the order of operators we get a negative sign, so α, β, γ must be order cyclically.

Q5. Since U is a 3×3 orthogonal matrix it has a cubic characteristic equation with real coefficients. From the fundamental theorem of algebra there must be three roots, one of which is real and the other two are complex conjugates of each other.

Q6.

a. We know that $\sigma_y^2 = 1$ where 1 is the identity matrix, so have that

$$(i\sigma_y\theta)^{2n} = (-1)^n\theta^{2n} \tag{22}$$
$$(i\sigma_y\theta)^{2n+1} = i(-1)^n\sigma_y\theta^{2n+1}. \tag{23}$$

Therefore

$$e^{i\sigma_y\theta} = \sum_{n=0}^{\infty}\frac{(i\sigma_y\theta)^n}{n!} \tag{24}$$
$$= \sum_{n=0}^{\infty}\frac{(i\sigma_y\theta)^{2n}}{2n!} + \sum_{n=0}^{\infty}\frac{(i\sigma_y\theta)^{2n+1}}{(2n+1)!} \tag{25}$$
$$= \sum_{n=0}^{\infty}\frac{(-\theta)^{2n}}{2n!} + i\sigma_y\sum_{n=0}^{\infty}\frac{(-\theta)^{2n+1}}{(2n+1)!} \tag{26}$$
$$= \cos\theta + i\sigma_y\sin\theta \tag{27}$$

b. As the a unitary matrix U must leave the norm of a vector invariant. So the eigenvalues of a unitary matrix must have the form $e^{i\theta_i}$, where i enumerates the different eigenvalues. Therefore if we diagonalize U by the similarity transformation $V^\dagger UV$ we have $e^{i\theta_j}$ along diagonal. So if we choose the hermitian operator to be $H_{ij} = \theta_i\delta_{ij}$ then clearly in the new basis $U = e^{iH}$. So in the old basis if we find $U = e^{VHV^\dagger}$. So every unitary matrix can be write as the exponential of a hermitian operator. The second part of question we did on HW2.

Q7.

1. As L_z is diagonal we are in the L_z basis and the eigenvalues of $L_z = 1, 0, -1$ are the physically measurable values.

2. As $L_z = 1$ state is

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

So the expectation values of L_x, L_x^2 are

$$\langle L_x\rangle = (1,0,0)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \tag{29}$$
$$\langle L_x^2\rangle = (1,0,0)\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \tag{30}$$

Therefore we have

$$\Delta L_x = \sqrt{\langle L_x^2\rangle - \langle L_x\rangle^2} \tag{31}$$
$$= \frac{1}{\sqrt{2}} \tag{32}$$

3. We have the characteristic equation of L_x

$$\omega(1-\omega^2) = 0 \tag{33}$$

so the eigenvalues of L_x are $1, 0, -1$. The eigenvector for $L_x = 1$ we find by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{34}$$

$$\Rightarrow a = \frac{b}{\sqrt{2}} = c \tag{35}$$

$$\Rightarrow |L_x = 1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{36}$$

Similarly we have

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{37}$$

$$\Rightarrow a = -\frac{b}{\sqrt{2}} = c \tag{38}$$

$$\Rightarrow |L_x = -1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \tag{39}$$

$$\tag{40}$$

and

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{41}$$

$$\Rightarrow a = -c \tag{42}$$

$$\Rightarrow b = 0 \tag{43}$$

$$\Rightarrow |L_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{44}$$

4. Clearly the only possible values of L_x are $1, 0, -1$ and to find the probabilities we expand $|L_z = -1\rangle$ in the L_x basis to find

$$|L_z = -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{45}$$

$$= \frac{1}{2}\left(|L_x = 1\rangle + |L_x = -1\rangle\right) - \frac{1}{\sqrt{2}}|L_x = 0\rangle \tag{46}$$

So the probabilities are $P(L_x = 1) = \frac{1}{4} = P(L_x = -1)$ and $P(L_x = 0) = \frac{1}{2}$.

5. We have the initial state is

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix} \tag{47}$$

and a measurement of L_z^2 is performed to find that it is 1. Now using the idea on p. 124 of Shankar we have

$$|\psi\rangle \rightarrow \frac{\mathcal{P}|\psi\rangle}{\langle \mathcal{P}\psi|\mathcal{P}|\psi\rangle} \tag{48}$$

where $\mathcal{P}$ is the projection operator. For the $L_z^2 = 1$ the projection operator has two pieces due to $L_z = \pm 1$. So we have

$$\mathcal{P} = |1\rangle\langle 1| + |-1\rangle\langle -1| \tag{49}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

So we find that

$$\mathcal{P}|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix} \tag{51}$$

$$= \begin{pmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}. \tag{52}$$

So the state after measurement is then

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix} \tag{53}$$

The probability that $L_z^2 = 1$ is $3/4$, now if L_z was measured we would find that $P(L_z = 1) = 1/4, P(L_z = 1) = 1/4, P(L_z = 0) = 0$

6. We have that

$$|\psi\rangle = \frac{1}{2} \begin{pmatrix} e^{i\delta_1} \\ \sqrt{2}e^{i\delta_2} \\ e^{i\delta_3} \end{pmatrix} \tag{54}$$

Therefore the probability of being in the $L_x = 0$ state is

$$|\langle L_x = 0|\psi\rangle|^2 = \frac{1}{8}|e^{i\delta_1} - e^{i\delta_3}|^2 \tag{55}$$

$$= \frac{1}{4}(1 - \cos(\delta_3 - \delta_1)). \tag{56}$$

So clearly the the probabilities depend on the relative phases. The overall phase is physically unimportant, but the relative phases have physical significance.