Problem Set 3

Physics 234 Spring Quarter 2004 H.J.Frisch HEP320 (702-7479)
April 12, 2004 Office Hours:Wed, Fri after class
(Other times possible- please call before coming over).

Due Monday, April 19 (hand in at class).

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Reading: *The Mystery of the Quantum Cakes*, by P. Kwiat and L. Hardy (class handout), and Shankar, Chapter 4, Sections 4.1 and 4.2. If there are sections of Chapter 1 that still bother you, please review them as well, and/or come and ask questions (being comfortable with Hermitian matrices and unitary transformations is critical).

Problems:

1. Prove that the matrix Ω is diagonalized by the matrix U formed of the eigenvectors of Ω (in columns) in the *similarity transformation*[1] $U^{-1}\Omega U$.

2. The trace of an $n \times n$ matrix A is defined as Trace $A = a_{11} + a_{22} + a_{33} + ...a_{nn}$.
   Let $|u>$ and $|v>$ be two vectors of finite norm. Show that Trace $|v><u| =< v|u>$.

3. Let H be a positive-definite Hermitian operator. Show that Trace H > 0.

4. Matrices as 'vectors':

   (a) Show that the set of all $n \times n$ matrices forms a linear vector space of dimension n^2.
   (b) Show that if A and B are two matrices, then trace $(A^\dagger B)$ satisfies all the requirements of the usual scalar product.
   (c) For $n = 2$, show explicitly that the four Pauli matrices
   $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
   (d) form an orthogonal basis. Prove that $\sigma_\alpha \sigma_\beta = i\sigma_\gamma$, α, β, γ cyclic.

5. Show that at least one of the eigenvalues of a 3×3 orthogonal matrix must be real, and that the other two are complex conjugates.

6. If a function f(x) has a power series expansion, the corresponding function f(A) for a matrix A is defined as the corresponding series in powers of the matrix, provided this series converges.
   (a) Prove that $e^{i\sigma_y\theta} = cos(\theta) + i\sigma_y sin(\theta)$.
   (b) Use the fact that any unitary matrix can be diagonalized by a suitable unitary transformation to show that any unitary operator U can be expressed as $U = e^{iH}$, where H is an Hermitian operator. Conversely, show that if H is Hermitian, U is unitary.

7. Shankar Exercise 4.2.1 (p129).

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[1] I discovered the perfectly useless but marvelous fact that this is also called an *Ähnlichkeitstransformation*.