PHYSICS 234 HOMEWORK 2 SOLUTIONS

1.8.1. The matrix we have to diagonalize is

$$\Omega = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix} \tag{1}$$

So the characteristic equation is

$$(1-\omega)((2-\omega)(4-\omega)-0) = 0. \tag{2}$$

So the eigenvalues are $1, 2, 4$. To find the eigenvector for $\omega = 1$ we have

$$\begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{3}$$

$$\Rightarrow b = 0 = c \tag{4}$$

So the eigenvector is

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

Similarly for $\omega = 2$ we have

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{6}$$

$$\Rightarrow \begin{cases} -a + 3b + c = 0 \\ b + 2c = 0 \end{cases} \tag{7}$$

$$\Rightarrow |2\rangle = \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}. \tag{8}$$

Finally the eigenvector corresponding to $\omega = 4$ is found from

$$\begin{pmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{9}$$

$$\Rightarrow -3a + c = 0 \tag{10}$$

$$\Rightarrow |4\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}. \tag{11}$$

This matrix is not hermitian and so as we can see the eigenvectors are not orthogonal as would be the case if the matrix was hermitian.

1.8.2. The matrix we are given is

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{12}$$

which is clearly hermitian. From looking at the matrix it should be clear that the eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{13}$$

whose corresponding eigenvalues are $1, -1, 0$ respectively. Let us choose U to be

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \tag{14}$$

So we have

$$U^\dagger \Omega U = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \tag{15}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{17}$$

1.8.3. We have

$$\Omega = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \tag{18}$$

Therefore the characteristic equation is

$$(2 - 2\omega)((3 - 2\omega)^2 - 1) = 0 \tag{19}$$

$$\Rightarrow (1 - \omega)^2(2 - \omega) = 0 \tag{20}$$

So we find the eigenvalues are $1, 1, 2$. To find the eigenvector corresponding to $\omega = 2$ we have

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & -1/2 \\ 0 & -1/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{21}$$

$$\Rightarrow c = -b \tag{22}$$

$$\Rightarrow a = 0 \tag{23}$$

$$\Rightarrow |2\rangle = \frac{1}{\sqrt{2b^2}}\begin{pmatrix} 0 \\ b \\ -b \end{pmatrix}. \tag{24}$$

Now the 2D space which works which corresponds the degenerate values of $\omega = 1$ is has to be perpendicular to this vector. So clearly the vectors in this space have the form

$$\frac{1}{\sqrt{a^2 + 2c^2}} \begin{pmatrix} a \\ c \\ -c \end{pmatrix}. \tag{25}$$

1.8.4. We have

$$\Omega = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \tag{26}$$

Therefore the characteristic equation is

$$\begin{aligned} ((4-\omega)(2-\omega)+1) &= 0 \qquad (27) \\ \Rightarrow (3-\omega)^2 &= 0. \qquad (28) \end{aligned}$$

So the only eigenvalue is $\omega = 3$ and therefore to find the corresponding eigenvectors we gave

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{29}$$

$$\Rightarrow b = -a \tag{30}$$

$$\Rightarrow |2\rangle = \frac{1}{\sqrt{2a^2}} \begin{pmatrix} a \\ -a \end{pmatrix}. \tag{31}$$

1.8.5. We have

$$\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{32}$$

and we see that

$$\Omega^\dagger = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{33}$$

Therefore we find

$$\begin{aligned} \Omega^\dagger\Omega &= \begin{pmatrix} \cos\theta & \sin -\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (34) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (35) \end{aligned}$$

The characteristic equation is

$$\begin{aligned} & (\cos\theta - \omega)^2 + \sin^2\theta = 0 \qquad (36) \\ \Rightarrow \quad & \omega^2 - 2\omega\cos\theta + 1 = 0 \qquad (37) \\ \Rightarrow \quad & \omega = \cos\theta \pm i\sin\theta = e^{\pm i\theta}. \qquad (38) \end{aligned}$$

To find the eigenvectors we have

$$\begin{pmatrix} \mp i\sin\theta & \sin\theta \\ -\sin\theta & \mp\sin\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{39}$$

$$\Rightarrow b = \pm ia. \tag{40}$$

So the eigenvectors are

$$|\omega = e^{\pm i\theta}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = 0 \tag{41}$$

So the transformation matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{42}$$

and therefore we have

$$\begin{aligned} U^\dagger \Omega U &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} && (43) \\ &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos\theta + i\sin\theta & \cos\theta - i\sin\theta \\ -\sin\theta + i\cos\theta & -\sin\theta - i\cos\theta \end{pmatrix} && (44) \\ &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} && (45) \end{aligned}$$

1.8.6. If Ω hermitian or unitary it can always be diagonalized so that the eiganvalues are the only non-trivial components along the diagonal of the matrix. The determinant of any diagonal matrix is the product of the diagonal elements, so the determinant of a unitary or hermitian matrix is the product of its eigenvalues or

$$Det(\Omega) = \prod_{i=1}^{n} \omega_i \tag{46}$$

Similarly as the trace is invariant under a unitary transformation and the trace of a any matrix is the sum of the diagonal elements we have

$$Tr(\Omega) = \sum_{i=1}^{n} \omega_i \tag{47}$$

1.8.7. We have

$$\Omega = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \tag{48}$$

so $Det(\Omega) = \omega_1\omega_2 = -3$ and $Tr(\Omega) = \omega_1 + \omega_2 = 2$. Using these two equations we have

$$\begin{aligned} &\omega^2 - 2\omega - 3 = 0 && (49) \\ &\Rightarrow \omega = 1 \text{ or } 2 && (50) \end{aligned}$$

If we calculate the characteristic equation we get the same quadratic equation as eqn.(50).

1.8.8. We have four hermitian matrices M^1, M^2, M^3, M^4 which satisfy the relation

$$M^i M^j + M^j M^i = 2\delta^{ij} I \tag{51}$$

Now if we set $i = j$ in eqn.(51) we have $(M^i)^2 = I$ so if we are in the diagonal basis of M^i we know that the square of the eigenvalues must be 1. As M^i is hermitian the eigenvalues have to be real so M^i has eigenvalues ± 1.

For $i \neq j$ we have $M^i M^j = -M^j M^i$ from eqn.(51). Now if we multiply this equation by M^i on the left and take the trace of both sides we get

$$\begin{aligned} Tr((M^i)^2 M^j) &= -Tr(M^i M^j M^i) && (52) \\ \Rightarrow Tr(M^j) &= -Tr(M^j (M^i)^2) && (53) \\ Tr(M^j) &= -Tr(M^j), && (54) \end{aligned}$$

where we have used the fact that $(M^i)^2 = I$ and the cyclic property of the trace.

Now as the eigenvalues are ± 1 and they are traceless we have that there must an equal number of 1 as -1. Therefore the M^i must be of even dimension.

1.8.9. From the equation for angular momentum we have

$$L_i = \sum_\alpha m_\alpha (\mathbf{r}_\alpha \times \mathbf{v}_\alpha)_i \tag{55}$$

where $\mathbf{v}_\alpha = \omega \times \mathbf{r}_\alpha$. So if we find that

$$\begin{aligned}
(\mathbf{r}_\alpha \times \mathbf{v}_\alpha)_i &= (\mathbf{r}_\alpha \times \omega \times \mathbf{r}_\alpha)_i && (56)\\
&= (r_\alpha)^2 \omega_i - (\mathbf{r}_\alpha \cdot \omega) r_{i\alpha} && (57)\\
&= \sum_j ((r_\alpha)^2 \delta_{ij} - r_{i\alpha} r_{j\alpha}) \omega_j . && (58)
\end{aligned}$$

So substituting into eqn.(55) we have

$$L_i = \sum_j M_{ij} \omega_j \tag{59}$$

where $M_{ij} = \sum_\alpha m_\alpha ((r_\alpha)^2 \delta_{ij} - r_{i\alpha} r_{j\alpha})$. Clearly from this equation angular momentum and angular velocity are not always parallel. Also M_{ij} is real and symmetric so it trivially hermitian and therefore it can be diagonalized to with three orthonormal eigenvectors. If the angular velocity points in one of these eigenvector direction then the angular momentum will also be parallel to it. Therefore there are atleast three directions in which the angular momentum and angular velocity are parallel.

For a sphere the moment of inertia is proportional to the identity matrix as this is the only matrix that has every vector as an eigenvector. Therefore the eigenvalues of M the sphere are all equal.

1.8.10. We have

$$\begin{aligned}
\Omega &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} && (60)\\
\Lambda &= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} && (61)
\end{aligned}$$

so the commutator of these two matrices is $[\Omega, \Lambda] = 0$ and therefore the two matrices can be simultaneously be diagonalized. Now as the question tells us that Ω is degenerate while Λ is nondegenerate. So the obvious thing to do is to first diagonalize Λ. The characteristic equation for Λ is

$$\begin{aligned}
(2-\omega)(\omega(\omega-2)-1) - (3-\omega) + (\omega - 1) &= 0 && (62)\\
\Rightarrow (2-\omega)(\omega^2 - 2\omega - 3) &= 0 && (63)\\
\Rightarrow \omega &= -1, 2 \text{ or } 3 && (64)
\end{aligned}$$

So for $\omega = -1$ we have

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{65}$$

$$\Rightarrow 3a + b + c = 0 \tag{66}$$

$$\Rightarrow a + b - c = 0 \tag{67}$$

$$\Rightarrow a - b + 3c = 0 \tag{68}$$

$$\Rightarrow c = -a \tag{69}$$

$$\Rightarrow b = -2a \tag{70}$$

$$\Rightarrow |-1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}. \tag{71}$$

Similarly for $\omega = 2$ we have

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{72}$$

$$\Rightarrow b + c = 0 \tag{73}$$

$$\Rightarrow a - 2b - c = 0 \tag{74}$$

$$\Rightarrow a - b = 0 \tag{75}$$

$$\Rightarrow c = -b \tag{76}$$

$$\Rightarrow a = b \tag{77}$$

$$\Rightarrow |2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \tag{78}$$

and for $\omega = 3$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{79}$$

$$\Rightarrow -a + b + c = 0 \tag{80}$$

$$\Rightarrow a - 3b - c = 0 \tag{81}$$

$$\Rightarrow a - b - c = 0 \tag{82}$$

$$\Rightarrow a = c \tag{83}$$

$$\Rightarrow b = 0 \tag{84}$$

$$\Rightarrow |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{85}$$

So transformation matrix is

$$U = \frac{1}{6} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ -2 & \sqrt{2} & 0 \\ -1 & -\sqrt{2} & \sqrt{3} \end{pmatrix} \tag{86}$$

in which case we find that

$$U^\dagger \Omega U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{87}$$

$$U^\dagger \Lambda U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{88}$$

1.8.11. The initial state is

$$|x(0)\rangle = \frac{1}{\sqrt{2}}(|I\rangle + |II\rangle). \tag{89}$$

So the state at any time t is

$$|x(t)\rangle = U(t)|x(0)\rangle \tag{90}$$

$$= (|I\rangle\langle I| \cos\omega_I t + |II\rangle\langle II| \cos\omega_{II} t)|x(0)\rangle \tag{91}$$

$$= \frac{1}{\sqrt{2}}(\cos\omega_I t|I\rangle + \cos\omega_{II} t|II\rangle) \tag{92}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\omega_I t + \cos\omega_{II} t \\ \cos\omega_I t - \cos\omega_{II} t \end{pmatrix} \tag{93}$$

which is exactly what we get in 1.8.39 when we use the same value of $|x(0)\rangle$.

1.8.12. We have the two equations

$$|\ddot{x}(t)\rangle = \Omega|x(t)\rangle \tag{94}$$

$$|x(t)\rangle = U(t)|x(0)\rangle \tag{95}$$

From eqn.(95) we also know that

$$|\ddot{x}(t)\rangle = \ddot{U}(t)|x(0)\rangle \tag{96}$$

and substituting eqn.(95) and eqn.(96) into eqn.(94) we find that

$$\ddot{U}(t) = \Omega U(t). \tag{97}$$

is the require equation of U. Now if we assume that both U and Ω are simultaneously diagonalizable then the only non-zero elements of U are the components U_{11} and U_{22} which have the following equations

$$\ddot{U}_{11}(t) = -\omega_1^2 U_{11}(t) \tag{98}$$

$$\ddot{U}_{22}(t) = -\omega_2^2 U_{22}(t). \tag{99}$$

Now if we have the initial condition that $|\dot{x}(t)\rangle = 0$ then $\dot{U}_{11} = 0\dot{U}_{22}$. In this case we find that $U_{11} = \cos\omega_1 t$ and $U_{22} = \cos\omega_2 t$ which is exactly corresponds to equation 1.8.43

1.9.1. As Ω is hermitian we can always diagonalize it. So a n product of Ω's will have the eigenvalues raised to the nth power along diagonal while all off diagonal elements will be zero. Therefore

$$f(\Omega) = \begin{pmatrix} f(\omega_1) & 0 & 0 \\ 0 & f(\omega_2) & 0 \\ 0 & 0 & \ddots \end{pmatrix}, \tag{100}$$

so the $f(\Omega) = \frac{1}{1-\Omega}$ is holds when each of the eigenvalues $\omega_i < 1$

1.9.2. We that H is hermitian, so $H^\dagger = H$. Now $U = e^{iH}$ so that $U^\dagger = e^{-iH}$. Now as H commutes with itself we can use the identity $e^A e^B = e^{A+B}$ we have that $U^\dagger U = 1$. So U is unitary.

1.9.3. Again if we diagonalize H and then evaluate the exponential then U is a diagonal matrix that has exponential of $i\omega_j$ as its jj entry. Therefore determinant of U corresponds to a product of these diagonal elements which due to the property of exponents leads to the exponential of the sum of all eigenvalues. Therefore $Det(U) = e^{i\,Tr(H)}$.

1.10.1. The important property of the δ function is

$$\int \delta(x)dx = 1 \tag{101}$$

therefore we have that

$$\int \delta(kx)dx = \int \delta(y)\frac{1}{|k|}dy \tag{102}$$

where we change variables from x to $y = kx$ and the absolute value is because the intergral is independent of the sign of x. As x and y are dummy integration indices we can relabel y with x and so $\delta(kx) = \frac{1}{|k|}\delta(x)$

1.10.2. Let us again use the integral of the δ function

$$\int \delta(f(x))dx = \int \delta(y)\frac{1}{|\frac{df(x_0)}{dx}|}dy \tag{103}$$

where we changed variables from x to $y = f(x)$ and then used the fact that the delta function is non-zero only at x_0 along with the fact that the sign does not make a difference. So we have $\delta(f(x)) = \frac{\delta(x)}{|\frac{df(x_0)}{dx}|}$

1.10.3. Again we see that

$$\frac{d}{dx}\theta(x-x') = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases} \tag{104}$$

as $\theta(x-x')$ is constant at all points except $x = x'$ at which there is a skip in the function so the derivative is infinite at this point. Finally the we show that

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d}{dx}\theta(x-x')dx = \theta(\epsilon) - \theta(-\epsilon) = 1 \tag{105}$$

Therefore $\frac{d}{dx}\theta(x-x') = \delta(x-x')$.

1.10.4. All we have to in his question is to calculate

$$\langle m|\psi(0)\rangle = \sqrt{\frac{2}{L}}\int_0^L \sin(\frac{m\pi x}{L})\psi(x,0)dx \tag{106}$$

where

$$\psi(x,0) = \begin{cases} \frac{2xh}{L} & 0 \leq x \leq \frac{L}{2} \\ \frac{2h}{L}(L-x) & \frac{L}{2} \leq x \leq L \end{cases} \tag{107}$$

So we find

$$\langle m|\psi(0)\rangle = \sqrt{\frac{2}{L}}\frac{2h}{L}\left(\int_0^{L/2} x\sin\left(\frac{m\pi x}{L}\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{m\pi x}{L}\right)dx\right) \quad (108)$$

$$= \sqrt{\frac{2}{L}}\frac{2h}{L}(1+(-1)^{m+1})\int_0^{L/2} x\sin\left(\frac{m\pi x}{L}\right)dx \quad (109)$$

$$= \sqrt{\frac{2}{L}}\frac{2h}{L}(1+(-1)^{m+1})\frac{L^2}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right) \quad (110)$$

$$= \sqrt{\frac{2}{L}}\frac{4hL}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right). \quad (111)$$

Now substituting into 1.10.59 we have

$$\psi(x,t) = \sum_{m=1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t\langle m|\psi(0)\rangle \quad (112)$$

$$= \sum_{m=1}^{\infty}\frac{8h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{m\pi x}{L}\right)\cos\omega_m t \quad (113)$$