PHYSICS 234 MIDTERM EXAM SOLUTIONS

Q1.

a). The spin matrix in the x-direction is

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

and the state given to us was

$$|\psi\rangle = e^{i\phi}\begin{pmatrix} \sqrt{\frac{4}{5}} \\ \sqrt{\frac{1}{5}} \end{pmatrix}. \tag{2}$$

So the expectation value of x-component of the spin is $\frac{2\hbar}{5}$.

b). In this question we do not care about the energy of the particle. All that is really wanted is first excited state. Since the box is between $\{-L/2, L/2\}$ the first excited state is

$$\psi(x) = A\sin\left(\frac{2\pi x}{L}\right). \tag{3}$$

Therefore the probability density is

$$P(x) = |\psi(x)|^2 = A^2\sin^2\left(\frac{2\pi x}{L}\right) \tag{4}$$

and the probability density is maximum when $\frac{dP}{dx} = 0$ or when

$$\begin{aligned}
2\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right) &= 0 && (5)\\
\Rightarrow \sin\left(\frac{4\pi x}{L}\right) &= 0 && (6)\\
\Rightarrow \frac{4\pi x}{L} &= n\pi && (7)\\
\Rightarrow x &= \frac{nL}{4}. && (8)
\end{aligned}$$

Since we know at n = 0 sin is zero the only other possible solutions are $\pm L/4$.

c). The potential is given to us as

$$V(x) = \begin{cases} V_o & x < 0 \\ 0 & x > 0 \end{cases} \tag{9}$$

so the wavefunction is then

$$\psi(x) = \begin{cases} Ae^{ipx} + Be^{-ipx} & x < 0 \\ Ce^{iqx} & x > 0 \end{cases} \tag{10}$$

where $p = \sqrt{\frac{2m(E-V_o)}{\hbar^2}}$ and $q = \sqrt{\frac{2mE}{\hbar^2}}$ and the $D = 0$ as the wave is coming from the left. So that the wave function be continuous and differentiable we have

$$A + B = C \tag{11}$$

$$p(A - B) = qC \tag{12}$$

$$\Rightarrow C = \frac{2p}{p+q}A. \tag{13}$$

So all that was required was this expression. So from this expression we can say the amplitude decrease across the boundary. The wave length is $\lambda = 2\pi/k$ and as $p < q$ we have that the initial wavelenght is greater than the final wave length.

Q4. Clearly the overlap integral is 0 as these are orthogonal states.

Q2.

a). The time-independent Schrodinger equation is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{14}$$

b). For $x < 0$ we have $V = 0$ and there the Schrodinger equation becomes

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x) \tag{15}$$

whose solution is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{16}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$.

c). For $x > 0$ we have $V = V_o$ with $E < V_o$ and there the Schrodinger equation becomes

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_o\psi(x) = E\psi(x) \tag{17}$$

whose solution is

$$\psi(x) = Ce^{-\kappa x} \tag{18}$$

where $\kappa = \sqrt{\frac{2m(V_o-E)}{\hbar^2}}$.

d). So we find the time-dependent wavefunction for the particle with energy E is

$$\psi(x) = \begin{cases} (Ae^{ikx} + Be^{-ikx})e^{-iEt/\hbar} & x < 0 \\ Ce^{-\kappa x}e^{-iEt/\hbar} & x > 0 \end{cases} \tag{19}$$

e). The boundary conditions at $x = 0$ give us the following equations

$$A + B = C \tag{20}$$

$$ik(A - B) = \kappa C. \tag{21}$$

f). Solving the boundary conditions give us the equations

$$A = \frac{1}{2}\left(1 - \frac{i\kappa}{k}\right) \tag{22}$$

$$B = \frac{1}{2}\left(1 + \frac{i\kappa}{k}\right) \tag{23}$$

g). Now in general a standing wave is formed when the two waves of equal amplitude, but moving in opposite directions interfere in the same region of space. So in the case of waves reflected from potential we need the reflection coefficient to be 1. In our question we know the reflected coefficient is

$$R = \frac{|B|^2}{|A|^2} = \frac{|1+\frac{i\kappa}{k}|^2}{|1-\frac{i\kappa}{k}|^2} \tag{24}$$

$$= 1 \tag{25}$$

h). The probability density is defined to be

$$P(x) = |\psi(x)|^2 \tag{26}$$

so for $x < 0$ the probability density is

$$P(x) = |A|^2 + |B|^2 + AB * e^{2ikx} + A * Be^{-2ikx} \tag{27}$$

Q3. This question is similar to that on the homework, except that the state is time independent. We know

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \tag{28}$$

Now the position operator is

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \tag{29}$$

so the expectation value of x is

$$\langle x\rangle = \sqrt{\frac{\hbar}{8m\omega}}\left(\langle 2|a|3\rangle + \langle 3|a^\dagger|2\rangle\right) \tag{30}$$

$$= \sqrt{\frac{\hbar}{8m\omega}}2\sqrt{3} \tag{31}$$

$$= \sqrt{\frac{3\hbar}{2m\omega}} \tag{32}$$