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1. Obtain Eqs. (9.7 a and b) by differentiating Eqs. (9.8 a and b) with respect to time.

Ans: Differentiating Eq. 9.8 a, we have

$$0 = \frac{dl}{dt} = \frac{d}{dt}(\mu r^2 \dot\theta)$$
$$= 2\mu r \dot r \dot\theta + \mu r^2 \ddot\theta$$
$$= 2\mu r(\dot r \dot\theta + \tfrac{1}{2} r \ddot\theta)$$

that is, $\mu(r\ddot\theta + 2\dot r\dot\theta) = 0$, which is just Eq. 9.7 b.

For Eq. 9.8b, we get

$$0 = \frac{dE}{dt} = \frac{d}{dt}[\tfrac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) + U(r)]$$
$$= \mu\dot r\ddot r + \mu r\dot r\dot\theta^2 + \mu r^2\dot\theta\ddot\theta + \dot r\frac{dU(r)}{dr}$$
$$= \mu\dot r\ddot r + \mu r\dot r\dot\theta^2 + \mu r^2\dot\theta\ddot\theta - \dot r f(r)$$
$$= \mu\dot r\ddot r + \mu r\dot r\dot\theta^2 + \mu r\dot\theta\cdot(r\ddot\theta) - \dot r f(r)$$
$$= \mu\dot r\ddot r + \mu r\dot r\dot\theta^2 - \mu r\dot\theta\cdot(2\dot r\dot\theta) - \dot r f(r)$$
$$= \dot r(\mu\ddot r - \mu r\dot\theta^2 - f(r))$$

i.e. $\mu(\ddot r - r\dot\theta^2) = f(r)$, which is Eq. 9.7 a.

2. A particle of mass 50 g moves under an attractive central force of magnitude $4r^3$ dynes. The angular momentum is equal to 1,000 g·cm²/s.

a. Find the effective potential energy.

b. Indicate on a sketch of the effective potential the total energy for circular motion.

c. The radius of the particle's orbit varies between r_0 and $2r_0$. Find r_0.

Ans: a. The potential energy corresponding to the central force is

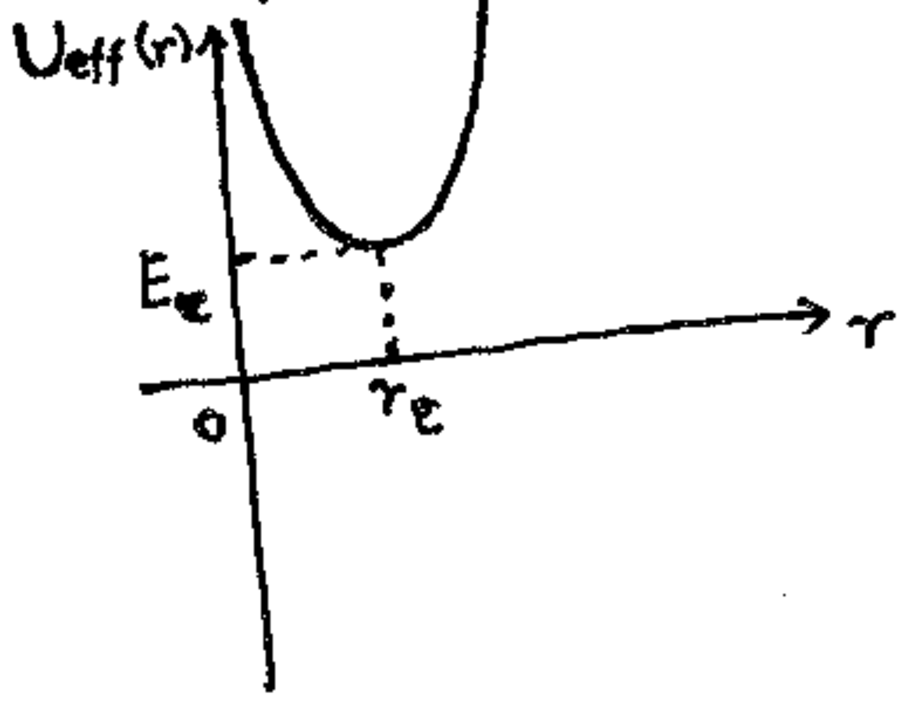
$$U(r) = -\int_0^r \vec f(\vec r)\cdot d\vec r$$
$$= \int_0^r 4r^3 \text{ dynes}\cdot dr$$
$$= r^4 \text{ dynes}\cdot\text{cm}$$

So the effective potential

$$U_{eff}(r) = U(r) + \frac{L^2}{2mr^2}$$
$$= \left(\frac{10^4}{r^2} + r^4\right) \text{ erg}$$

where r is in the unit of centimeters.

b. The effective potential looks like



The circular orbit appears at

$$\left.\frac{d\,U_{eff}(r)}{dr}\right|_{r=r_c} = \left(4r_c^3 - \frac{2\times10^4}{r_c^3}\right)\text{erg}$$
$$= 0$$

i.e. $r_c = 5^{1/6} \times 10^{1/2}$ (cm). The total energy for the circular orbit

$$E_c = U_{eff}(r_c)$$
$$= 3\,r_c^4 \text{ erg}$$
$$= 5^{2/3} \times 300 \text{ erg}$$
$$\approx 877 \text{ erg}$$

as shown in the sketch above.

c. The radius varies between r_0 and $2r_0$, which means

$$U_{eff}(r_0) = U_{eff}(2r_0)$$

that is,

$$\left(\frac{10^4}{r_0^2} + r_0^4\right)\text{erg} = \left(\frac{10^4}{4r_0^2} + 16\,r_0^4\right)\text{erg}$$

Therefore we have

$$r_0 = 500^{1/6} \text{ cm}$$
$$\approx 2.82 \text{ cm}$$

3. For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$, where $A > 0$?

Ans: The effective potential

$$U_{eff}(r) = \frac{L^2}{2mr^2} + U(r)$$
$$= \frac{L^2}{2mr^2} - \frac{A}{r^n}$$

For the circular orbit with radius r_c, we have

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_c} = -\frac{2L^2}{2mr_c^3} + \frac{nA}{r_c^{n+1}}$$
$$= 0$$

that is, $r_c = \left(n \cdot \frac{mA}{L^2}\right)^{\frac{1}{n-2}}$. Therefore

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_c} = \frac{3L^2}{mr_c^4} - \frac{n(n+1)A}{r_c^{n+2}}$$
$$= \frac{nA(2-n)}{r_c^{n+2}}$$

For the orbit to be stable, we require $\left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_c} > 0$, i.e. $0<n<2$.

4. A rocket is in elliptic orbit around the earth. To put it into an escape orbit, its engine is fired briefly, changing the rocket's velocity by $\Delta\vec{V}$. Where in the orbit, and in what direction, should the firing occur to obtain escape with a minimum value of ΔV?

Ans: To attain escape, the required additional energy is a constant which can be expressed as

$$\Delta E = \frac{1}{2}M(\vec{V}+\Delta\vec{V})^2 - \frac{1}{2}M\vec{V}^2$$

Hence

$$\Delta V \geq |\vec{V}+\Delta\vec{V}| - V$$
$$= \sqrt{\frac{2\Delta E}{M} + V^2} - V$$
$$= \frac{\frac{2\Delta E}{M}}{\sqrt{\frac{2\Delta E}{M}+V^2} + V}$$

i.e. to have the minimum ΔV we want the firing to occur when V is maximum, or equivalently, when the rocket is closest to the earth, and in the same direction with the velocity $\vec{V}$.

5. Halley's comet is in an elliptic orbit about the sun. The eccentricity of the orbit is 0.967 and the period is 76 years. The mass of the sun is 2×10^{30} kg, and $G = 6.67 \times 10^{-11}\ N\cdot m^2/kg^2$.

a. Using these data, determine the distance of Halley's comet from the sun at perihelion and at aphelion.

b. What is the speed of Halley's comet when it is closest to the sun?

Ans: a. According to Kepler's 3rd law, the semimajor axis of the comet's orbit is

$$a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}$$

$$= \left(\frac{6.67 \times 10^{-11}\ N\cdot m^2/kg^2 \cdot 2\times 10^{30}\ kg \cdot (76\ yrs)^2}{4\pi^2}\right)^{1/3}$$

$$= 2.69 \times 10^{12}\ m$$

The distance of the comet from the sun at perihelion

$$d_p = (1-e)\cdot a$$

$$= 8.9 \times 10^{10}\ m$$

and at aphelion

$$d_a = (1+e)\cdot a$$

$$= 5.3 \times 10^{12}\ m$$

b. Kepler's 2nd law tells the area swept through by the comet per unit time is a constant

$$\frac{dS}{dt} = \frac{S}{T} = \frac{\pi ab}{T} = \frac{\pi\sqrt{1-e^2}\,a^2}{T}$$

At perihelion, we have

$$\left.\frac{dS}{dt}\right|_{perihelion} = \frac{1}{2} v_p d_p$$

thus the speed

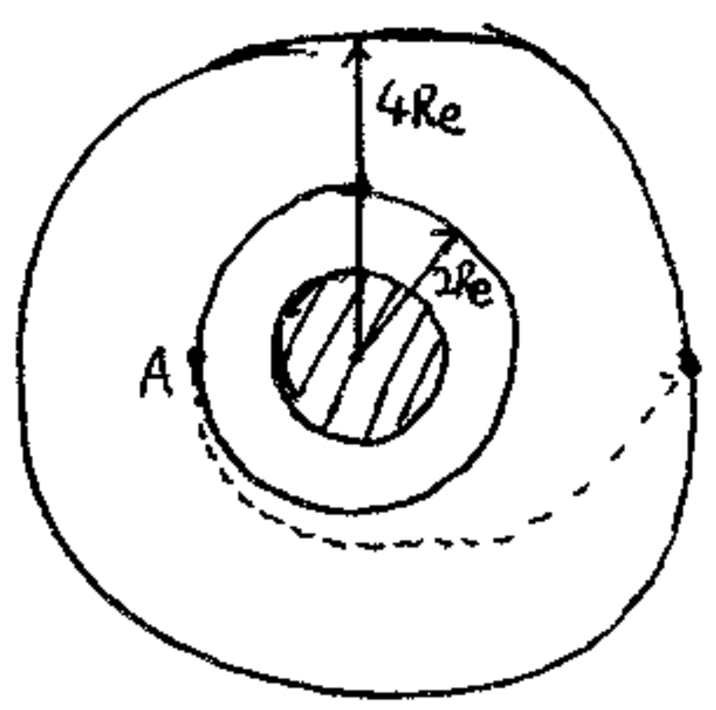
$$v_p = \frac{2\pi\sqrt{1-e^2}\,a^2}{T\cdot d_p}$$

$$= \frac{2\pi a}{T}\cdot\sqrt{\frac{1+e}{1-e}} = 5.4 \times 10^4\ m/s.$$

6. A space vehicle is in circular orbit about the earth. The mass of the vehicle is 3,000 kg and the radius of the orbit is $2R_e = 12{,}800$ km. It is desired to transfer the vehicle to a circular orbit of radius $4R_e$.

a. What is the minimum energy expenditure required for the transfer?

b. An effecient way to accomplish the transfer is to use a semielliptical orbit (known as a Hohmann transfer orbit), as shown. What velocity changes are required at the points of intersection, A and B?



Ans: a. The energy of the vehicle before the transfer

$$E_1 = \frac{1}{2} m v_1^2 - \frac{GMm}{r_1}$$
$$= \frac{1}{2}\frac{m v_1^2}{r_1} \cdot r_1 - \frac{GMm}{r_1}$$
$$= \frac{1}{2}\frac{GMm}{r_1^2} \cdot r_1 - \frac{GMm}{r_1}$$
$$= -\frac{1}{2}\frac{GMm}{r_1}$$
$$= -\frac{GMm}{4R_e}$$

and the energy after the transfer

$$E_2 = -\frac{1}{2}\frac{GMm}{r_2}$$
$$= -\frac{GMm}{8R_e}$$

So the minimum energy required

$$\Delta E = E_2 - E_1$$
$$= \frac{GMm}{8R_e}$$
$$= 7.82 \times 10^{15}\ \mathrm{J}$$

b. The semi major axis of the Hohmann orbit is

$$a = \frac{1}{2}(r_1 + r_2)$$
$$= 3R_e$$

and the eccentricity

$$e = 1 - \frac{r_1}{a}$$
$$= \frac{1}{3}$$

So Kepler's 3rd law tells the period

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

thus the area swept through in unit time, according to Kepler's 2nd law, is

$$\frac{ds}{dt} = \frac{S}{T} = \frac{\pi ab}{T} = \sqrt{1-e^2}\,\frac{\pi a^2}{T}$$
$$= \frac{1}{2}\sqrt{1-e^2}\sqrt{GMa}$$

We also have

$$\frac{ds}{dt} = \frac{1}{2} r_1 v_A = \frac{1}{2} r_2 v_B$$

therefore

$$v_A = \frac{\sqrt{1-e^2}\sqrt{GMa}}{r_1} = \sqrt{\frac{2}{3}}\sqrt{\frac{GM}{Re}}$$

$$v_B = \frac{\sqrt{1-e^2}\sqrt{GMa}}{r_2} = \sqrt{\frac{GM}{6Re}}$$

Now the circular orbits satisfy

$$m\frac{v_1^2}{r_1} = \frac{GMm}{r_1^2}$$
$$m\frac{v_2^2}{r_2} = \frac{GMm}{r_2^2}$$

hence

$$v_1 = \sqrt{\frac{GM}{r_1}} = \sqrt{\frac{GM}{2Re}}$$
$$v_2 = \sqrt{\frac{GM}{r_2}} = \frac{1}{2}\sqrt{\frac{GM}{Re}}$$

The required velocity changes at A and B respectively, are

$$\Delta v_A = v_A - v_1 = \frac{2-\sqrt{3}}{\sqrt{6}}\sqrt{\frac{GM}{Re}} = 4.99 \times 10^5 \ m/s$$
$$\Delta v_B = v_2 - v_B = \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)\sqrt{\frac{GM}{Re}} = 4.19 \times 10^5 \ m/s$$

with the directions both parallel to the original velocities.

Note: I'm sorry that I made a serious mistake in the above calculations! The space vehicle is orbiting around the earth, while I thought it's around the sun. Therefore I should replace all the M's by $M_\oplus$ and get

$$\Delta E = \frac{G M_\oplus m}{8 R_e}$$

$$= 2.34 \times 10^{10}\ \mathrm{J}$$

$$\Delta v_A = \frac{2-\sqrt{3}}{\sqrt{6}} \sqrt{\frac{G M_\oplus}{R_e}}$$

$$= 864\ \mathrm{m/s}$$

$$\Delta v_B = \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right) \sqrt{\frac{G M_\oplus}{R_e}}$$

$$= 726\ \mathrm{m/s}$$