Physics 141
Solutions for Problem Set 8
Due Monday, November 24

Problem 1.
$m_1 = m_2 = 5$ kg. The strings and the pulleys are massless.

(a) Draw a coordinate system with labeled coordinates for weights and the movable pulley.

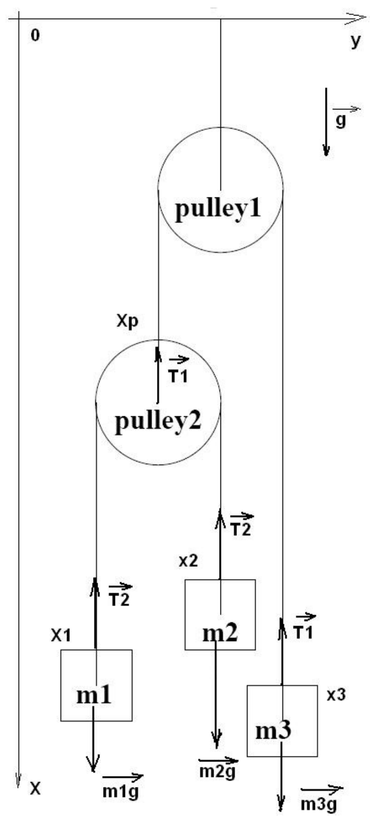

Figure 1: Two weights of 10 kg each hang from a string that is draped over a pulley, with one weight on each side. The pulley in turn hangs from a second string that is draped over a second pulley, connected on the other side to the third weight.

Variable x_i stands for the x-coordinate of mass m_i, where the index i can be 1,2, or 3.

(b) Write the equation for the length of the first string in term of the coordinates of the two weights tied to it.

$$l_1 = (x_1 - x_p) + (x_2 - x_p) + \pi \cdot R, \tag{1}$$

where R is the radius of the pulleys (it is constant).

(c) Write the equation for the length of the second string in term of the coordinates of the weight and the pulley tied to it.

$$l_2 = (x_3 - x_{p1}) + (x_p - x_{p1}) + \pi \cdot R, \tag{2}$$

where x_{p1} is the position of the 1st pulley (it is constant too).

(d) Find the equations of constraint on the accelerations by differentiating the equations of constraint on the positions of the weights.

$$2 \cdot a_p = a_1 + a_2 \tag{3}$$

$$a_p = -a_3 \tag{4}$$

(e) Set up the equations of motion for the 3 weights and the movable pulley.

Let's apply the second Newton's law to the three blocks and the first pulley:

$$m_1 \cdot a_1 = m_1 \cdot g - T_2 \tag{5}$$

$$m_2 \cdot a_2 = m_2 \cdot g - T_2 \tag{6}$$

$$m_3 \cdot a_1 = m_3 \cdot g - T_1 \tag{7}$$

Since the first pulley is massless:

$$0 = T_1 - 2T_2 \tag{8}$$

We obtain the following set of equations:

$$\begin{aligned} &m_1(a_1 - g) = m_2(a_2 - g) \\ &m_3(a_3 - g) = 2m_1(a_1 - g) \\ &a_1 + a_2 = -2a_3 \\ &a_p = -a_3 \end{aligned} \tag{9}$$

Problem 2 (6.39 from K&K)

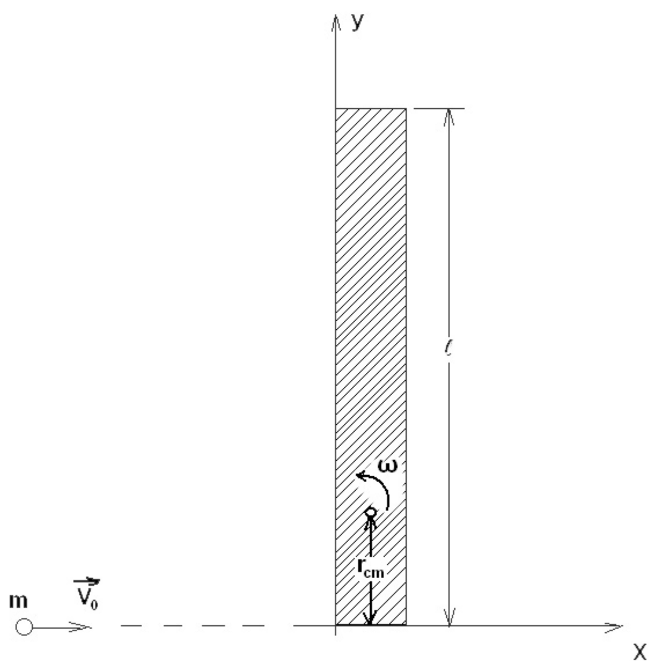

Figure 2: A boy of mass m runs on ice the with velocity $\vec{v}$ and steps on the end of a plink of length l and mass M which is perpendicular to his path.

a) The collision is inelastic. The plank and the boy will move together after the boy steps on the end of the plank. The system of the boy and the plank will be rotating around its center of mass with constant frequency. The center of mass will be moving with constant velocity $\vec{v}$. The velocity $\vec{v}$ is parallel to $\vec{v_0}$:

$$(M+m)\vec{v} = m\vec{v_0}. \tag{10}$$

b) Since the total angular momentum is conserved:

$$I\omega = mv_0 r_{cm}, \tag{11}$$

where r_{cm} is the position of the center of mass of the system "the plank + the boy", and I is the moment of inertia of the system. The position of the center of mass is

$$r_{cm}m = M(l/2 - r_{cm}). \tag{12}$$

The moment of of inertia is

$$I = mr_{cm}^2 + M(l/2 - r_{cm})^2 + Ml^2/12. \tag{13}$$

Let's define x_{rest} as the distance from the center of mass to the point which is at rest immediately after the collision:

$$\omega x_{rest} = v. \tag{14}$$

Therefore we obtain:

$$x_{rest} = \frac{I}{(M+m)r_{cm}} = \frac{lm}{2(M+m)} + \frac{l}{6} \tag{15}$$

and

$$x_{rest} + r_{cm} = \frac{2l}{3}. \tag{16}$$

The point on the plank which is at rest immediately after the collision is $2l/3$ from the boy.

Problem 3 (7.4 from K&K)

A torque caused by the reaction forces from the shaft and the surface, M_m, rotates the angular momentum of the millstone, L:

$$\frac{d}{dt}L = M_m, \tag{17}$$

where $dL/dt = L\Omega$ and $M_m = (N - Mg) * R$.

The angular momentum is

$$L = I * \omega, \tag{18}$$

where $I = Mb^2/2$ and $\Omega R = \omega b$.

By solving for N we obtain:

$$N = M(g + \frac{b\Omega^2}{2}). \tag{19}$$

Problem 4 (7.6 from K&K)

The angular momentum of the coin is rotated by the torque created by the reaction force N, force of friction F_{fr}. The reaction force negates the gravitation force mg exactly so that the coin has no vertical acceleration (i.e. $mg = N$). The force of friction creates the acceleration which causes the circular motion of the center of mass of the coin:

$$F_{fr} = \frac{mv^2}{R}. \tag{20}$$

Therefore the torque is

$$M = N \cdot b\sin(\phi) - F_{fr} \cdot b\cos(\phi) = mgb\sin(\phi) - \frac{mv^2\cos(\phi)}{R}. \quad (21)$$

$$M = \frac{d}{dt}L = L \cdot \Omega, \quad (22)$$

where $R\Omega = v$, $L = I\omega$ and $\omega = v/b$. The moment of inertia of the coin is $I = mb^2/2$.

We obtain

$$\frac{mv^2b}{2R}\cos(\phi) + \frac{mbv^2}{R}\cos(\phi) = mgb\sin(\phi) \quad (23)$$

and

$$\tan(\phi) = \frac{3v^2}{2Rg}. \quad (24)$$

Problem 5 (7.9 from K&K)

The problem is similar to the previous Problem 5. The torque rotating the angular momentum of the two wheels is created by the force of gravity Mg, the reaction force N, and the force of friction F_{fr}. Similarly to the previous problem we find that $N = mg$ and $F_{fr} = Mv^2/R$. The torque action on the axis of the wheels calculated around the center of mass is

$$M_w = 2l \cdot N \cdot \sin(\phi) - 2l \cdot F_{fr} \cdot \cos(\phi) = 2l \cdot Mg \cdot \sin(\phi) - 2\frac{lMV^2}{R}\cos(\phi). \quad (25)$$

$$\frac{d}{dt}L = M_w, \quad (26)$$

where $dL/dt = L\Omega$, $\Omega = V/R$, $L = 2I\omega$ (since the bicycle has 2 wheels), $\omega = V/l$, and $I = ml^2$ (by definition of the problem the moment of inertia of each wheel is ml^2).

Now we can calculate

$$\frac{2mlV^2}{R}\cos(\phi) = Mg2l\sin(\phi) - 2\frac{MV^2}{R}l\cos(\phi) \quad (27)$$

and

$$\tan(\phi) = \frac{V^2}{gR}\left(1 + \frac{m}{M}\right) \quad (28)$$

Problem 6 (7.11 from K&K)
A particle of mass m is located at (x, y, z) = (2, 0, 3).
a)Find its moments and products of inertia relative to the origin (Tensor of inertia).

$$\begin{aligned}
I_{xx} &= m(y^2 + z^2) = 9m \\
I_{yy} &= m(x^2 + z^2) = 13m \\
I_{zz} &= m(y^2 + x^2) = 4m \\
I_{xy} &= I_{yx} = -m \cdot x \cdot y = 0 \\
I_{xz} &= I_{zx} = -m \cdot x \cdot z = -6m \\
I_{zy} &= I_{yz} = -m \cdot z \cdot y = 0
\end{aligned} \tag{29}$$

b) The particle undergoes pure rotation about the Z axis through a small angle α. Show that its moments of inertia are unchanged to first order in α if $\alpha \ll 1$.

$$(x, y, z) \to (x', y', z') = (2\cos(\alpha), 2\sin(\alpha), 3) \approx (2, 2 \cdot \alpha, 3), \tag{30}$$

at the first order in α.

The new moments of inertia are

$$\begin{aligned}
I'_{xx} &\approx m(y'^2 + z'^2) = 9m + 4m \cdot \alpha^2 \approx 9m \\
I'_{yy} &\approx m(x'^2 + z'^2) = 13m \\
I'_{zz} &\approx m(y'^2 + x'^2) = 4m + 4m \cdot \alpha^2 \approx 4m
\end{aligned} \tag{31}$$

The moments of inertia are unchanged to first order in α.